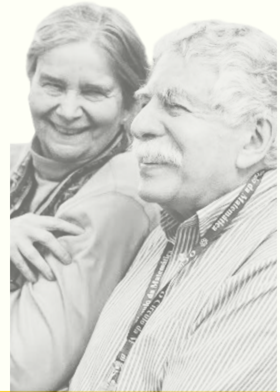
ESSENCE OF MATH CIRCLES



Discovery strengthens, instruction weakens.

Out of the Labyrinth

By Bob & Ellen Kaplan

How we aim to have everyone fall in reciprocated love with math works at every level of mathematical sophistication, so can be most clearly shown if we look at a classroom of four-to five-year-olds, new to The Math Circle — our Math Circle, that is, where students come up with insights together and invent proofs in collegial conversation. Not a hint of competition in the air to cheapen the profundity of this greatest of the arts.

Skip artificial introductions of “What’s your favorite color” and “Who’s your baseball hero”: they may be younger and shorter than you, but these are your colleagues in the adventures ahead, so talk from the first with them as you would to admired equals.

“Hello, my name is Leslie. You have three friends over to play and your mother has made a delicious sandwich for all of you to share, so she wants to cut it into equal parts — (pause, to check if they come up with ‘four’, just to make sure what page you’re all on) — four equal parts. How should she do it?”

No drawing of a sandwich or of a square: this is minds, not hands on, and the important drawings will come soon. You should get a lot of suggestions from the six to eight people there (not too many more, for a good conversation where you all get to know one another’s ways of thought). Welcome all offers, and now lightly sketch each (not geometric diagrams, but bread-like approximates). If cutting into squares doesn’t come up, nudge the conversation toward it. Important note: our approach isn’t that of Socrates or Moore: no eliciting of answers according to pre-ordained schema, but the free flow of invention and zaniness, with goals of your own kept in mind (these may change as the conversation takes unexpected turns).

Now that you have your drawing of a sandwich cut in squares, ask for other possible ways, aiming for quartering on the diagonals — if this hasn’t already come up. Once it has, draw the square and diagonal cut sandwiches again, away from the rest but lined up near each other, as alike in size as you can casually manage, and check with them that these are equally good drawings.

The critical moment has arrived: draw below the first a quarter square extracted from it, below the second a triangle — long side at the bottom. “This is the kind of sandwich you love — which piece would you rather have?” Almost inevitably, the triangle will be chosen — “because it’s bigger.” Math has now begun.

“Really? Let’s see — “ redraw each abstracted from the bread shape, now as geometric square and triangle (you could mumble something about “I’m just doing this to see it more clearly”, but a crucial digression is waiting for the right moment to happen, ideally when they bring the issue up: are we talking about bread or shape, real or pretend sandwiches?)

Ask again, and likely they’ll stick to their guns: the triangle is “bigger”, “there’s more of the sandwich there, obviously, than in the quarter square.” Ask how they know and then listen closely, writing up (more for yourself than them, since they likely can’t yet read) the idea behind each answer. From “it’s just obvious” to “the triangle’s side is longer”: you’re hearing how the eye, and spatial intuition, wrongly generalize from linear to planar measure.

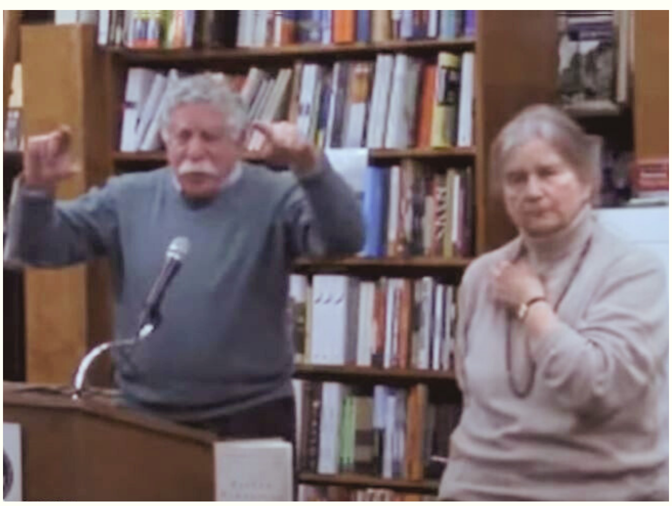
Attention, curiosity and feistiness are aroused: how, they ask, could you not see it? How could there be any doubt? Some come up to take the chalk from your hand, to make it clear. Riding this wave of intensity, either follow the conversation’s fall-line, or Pied Piper beckon them in this wrong, frustrating, but in the end very fruitful direction: “bigger”, “more of” — let’s figure out what we’re talking about: give it a name but not a definition, ‘area’.

Encourage the conversation to come up with tiling each shape with small unit squares (suggest it yourself, if they don’t): now an intractable geometric problem has been made arithmetic. Easy to tile the square and count the number, letting them take the lead again. But the triangle? A hideous jig-saw puzzle of truncated squares along the diagonals, scrambling now to piece them together into makeshift squares to finish the counting. Math has its glories but also its despairs, and we're just reached one. Perhaps it's hopeless; perhaps we won't be able to do it, perhaps no human could. Feints at the notion of approximation, slackened standards of accuracy, flights of fantasy, glimmer and pass. They now know what each of us has felt at a problem's midnight.

This isn't an hour's problem: it has no time limit, and may go on over the whole ten weeks of the course, as doggedness and attention-spans lengthen. Pick yourselves up, dust yourselves off, start all over again. You may need to take the lead (or willfully misinterpret a suggestion in the direction you want): let's scrap that line of thought and try another. What if we slide the quarter square over the triangle, base along base, until the square's right-hand vertical edge coincides with the triangle's altitude to its long base? (At this point, if it seems right, you might even hand them cut-outs to play with.) And now look: for looking leads to seeing.

Ah! The square now covers only half of the triangle, whose right-hand half lies exposed — and someone says: "I know this is wrong, but that exposed half matches the upper, empty triangle half of the square." Everyone talks at once: cut it off, rotate it, lift it up and put it there! It fits! This is crazy, but there's exactly as much of each! The glory of the move makes up for the dashed original certainty.

A great insight — but not the most elegant proof. Translation, dissection, rotation and translation again: can we keep the insight and simplify the road to it? Let them experiment,



discard, discover, and judge when to nudge them toward this approach: don't bother with the square, just cut the triangle along its altitude, and rotate to turn it into a square! Their mathematical sophistication will have grown during all their labors to appreciate this move, and the underlying comparing of the triangle to itself.

It's now, if not earlier, that the important digression we spoke of before may come up: slicing the bread loses some of it — the more, the duller or wider the knife-blade. How take this into account? A conversation now about the real objects of math, about abstracted shapes that leave the dough behind, about scissors sharper than knives, drawn lines thinner than cut lines, lines in the mind thinner yet. This will be only the first of many such conversations, a figured bass to all the music you will play together, so no need to press beyond present satisfaction.

Are we done? Far from it. We've gone from the false help offered by arithmetic to the more and more purely geometric, and are convinced. Yet a deeper proof, beyond the geometric, lies waiting to be born: a proof reduced to transparent logic. Recall that your mother wanted to make four equal parts for you and your friends to eat, and we've seen that she could have done this in two or more ways (perhaps vertical strips, for example, had come up in the course of suggestions). If each way ended with four equal parts, mustn't any one of them have as much in it as any other, no matter how the quarter-parts were made? Shape distracted us: a fourth of the whole remains a fourth of the whole, no matter how you slice it.

And now we're ready to go on in any of a number of directions: wavy cuts, three dimensions, back to understand area and tiling with unit squares, integration… the Bolyai-Gerwien Theorem, Dehn's Theorem, symmetries and motions and groups sparkle on the horizon.

We've said our approach isn't Socratic, yet time and again have spoken of enticing, suggesting, nudging the conversation. The students do the climbing, we sherpas bring up the supplies and may at dangerous moments point out crevasses; there is no fixed method here, all's a response to personalities and the character of the problem. One's grand flights, one's tootings at the weddings of the soul, occur, Wallace Stevens pointed out, when they occur, and can't be legislated for. A Math Circle is a high-wire act, its only safety net that woven by the trust and empathy forming from fluid exchanges.

Adventure!