Some parts of math are slippery, some are sheer. We have to learn the mountaineer's skill of clinging on *here* while stretching out to *there*. The ingredients are already in our human nature: stubbornness and its cousin, orneriness; a threshold of frustration that can be raised, a span of attention that can be lengthened, and holding on turned inside out: putting on hold.

Stubbornness

You have only to look at the Wright brothers' faces to know why they overcame the thousand obstacles between Dayton and Kitty Hawk: the mosquitoes, misleading tables, rain, wind, shattered spars. They hunkered down and inched forward again over much old and a little new ground.

In math it is often a case of the data piling up with no system to it, or a system emerging only to crumble at the next example. You come to a crossroads, and nothing in your intuition or in the landscape inclines you to follow one way rather than the other; if you try to follow both, your attention and energy drain away. In most other human callings, the terrain where you find yourself helps your moving through it, and general experience strengthens intuition—but it takes a lot of coming to know the invisible substance of math before you can make its alien-seeming abstraction a locale in your thought, and feel comfortable enough to stroll through it and begin to see that the streets have a plan and the buildings a code.

This sort of grasping is especially difficult because the frustrations in math are so definitive. They strike deep and happen often. It may be worse for novice chess players, who lose match after match before they begin to win—but at least they profit from each loss by seeing vividly that you need to look more than two (or three, or four . . .) moves ahead, or that certain tactics, good in themselves, suit ill with a particular strategy. You may have to lose the first fifty times you play Go before an inkling dawns of where the sinks and sources of power are on the board. In math, however, you're beaten not by a human master but by the uncommunicative subject itself. Say you're attempting to detect a pattern among Pythagorean triples (those integers x, y, and z such that $x^2 + y^2 = z^2$). One failed attempt may follow another with you being none, or ever so little, the wiser. Just when a pattern seems to settle in (two of the three must be consecutive integers, like 3, 4, 5; 7, 24, 25; or 20, 21, 29), a counterexample spoils it (28, 45, 53). You hear the problem saying that you are too stupid to find the pattern—or that of course there is none, and you were a fool to think there was. Ghosts of pattern spread out and disperse like spume on the restless sea.

How do you develop the kind of doggedness to get through these dark moments? A good teacher can certainly help, by setting a sequence of problems that build up confidence through increasing the resistance gradually. Fellow students help too: working conversationally with others shares out the frustrations and so lessens self-doubt. Both can make the play count for more than the winning or losing. Good, we've found one Pythagorean triple (say 3, 4, 5); and another: 6, 8, 10; ah—any multiple of a triple that works will work too—that's a real gain. And there are triples that aren't multiples of the one we found (20, 21, 29 isn't a multiple of 3, 4, 5): so something has come to light and there's no danger of being bored; more revelations must lie in wait. Dig a level deeper.

The kind of tenacious oblivion needed now has a negative and a positive source. Negatively, you cast yourself in the heroic mode of opposition, invoking your favorite figure of resistance. Dancers and athletes have a well-earned sense that you can still fail despite all your totally dedicated work—and that you then rub your bruises and begin again. Positively, you focus on the particulars of the handful of dirt you've just scooped up as you tunnel doggedly forward: this pebble in it, that fragment of china. So (3, 4, 5), (20, 21, 29), (7, 24, 25) and (28, 45, 53) are Pythagorean triples—well, an odd and an even to start with, then an odd. Nothing helpful here, perhaps, but bear it in mind.

Stubbornness roots you in the terrain. We were holding a Math Circle class at Microsoft, in Seattle, for the eight- to twelve-year-old children of some researchers there. The format was a casual lunch with sandwiches. "Has anyone heard of the Pythagorean Theorem?" we asked, while munching. Many had. "OK, would someone go up and draw a right triangle on the whiteboard . . . oh, and make its legs the same length." A confident twelve-year-old boy drew a large red right triangle.

"Swell. Let's say each of its legs is a unit long." He put a "1" by each of its legs. "So how long is the hypotenuse?" A pair of twins called out, almost together: "The square root of two!"

"Right. And what is the square root of two—I mean, what number does it turn out to be?"

Fortunately there wasn't a hand calculator in sight, so they were forced to reason it out. An eight-year-old girl said 1 was too small, since 1^2 was 1, and a ten-year-old said that 2 was too large, since 2^2 was 4. The likely candidate 3/2 proved to be just too big, giving 9/4 when squared. We had in mind luring them into a proof, after a few frustrations, that $\sqrt{2}$ couldn't be any fraction whatever—but that wasn't the direction thought was taking this blustery March afternoon. They were going to pin down the fraction hiding behind the mask of $\sqrt{2}$, and they were going to do it before lunch was over.

Round numbers are always false.—Samuel Johnson

The twins went to the board with blue markers in their hands and, at the suggestion of the oldest girl there, patiently multiplied 14/10 by itself: 196/100, so we were definitely just about finished. While ideas raced around the table about products of two negatives and why not try decimals and whether the final answer's numerator would be even or odd, the twins quietly calculated $(142/100)^2$ and then $(141/100)^2$, with results elating or depressing, depending on your point of view. They were very fond of multiplying, and as we started to bet what size fraction would finally do it, numbers appeared out of the blue on the white sky. They actually calculated

$$(1414/1000)^2 = 1999396/1000000,$$

and—while we held our breaths—

$$(1415/1000)^2 = 2002225/1000000.$$

Now that the hunt was all but over, others joined in the calculating frenzy, often darkening the waters with conflicting results.

We asked, in a pause which owed more to catching a second wind than to fatigue, what number the numerator would have to end with so that, when squaring the fraction, the result would be all zeroes after the 2.

A voice in the wilderness: "Five in one, six in the other?"

"But they have to be the same!"

"Then it can only end in zero."

"And before the zero?"

"Before the zero—another zero—"

It wasn't so much that lights went on as that the light went *off*.

"There's nothing that works . . . "

"We've multiplied wrong . . . "

The boy who, so long ago, had drawn the triangle on the board, went silently up and erased its hypotenuse.

"You mean . . . " we began.

"This triangle has no hypotenuse," he said—and with that the class ended. (We had exactly the same experience with a group of Scottish children at the other end of the economic scale, showing that mathematics is universal even in its entrées to error: we are all novices when it comes to thinking of numbers not as objects but as sequences.)

Did the class end in failure, do you think? It seems to us rather that their *stubbornness* gave them an ineradicable sense of the terrain. The next time that hypotenuse once again sits there gleaming evilly at them—a day from now for some, a month, a year for others—they'll have a vivid context in which to approach it, and the lay of the land they traveled over together may push thought toward a next stratum down in the search for how a line can have a length that isn't rational. Imagination is born from the vivid conviction that what can't be, must be.