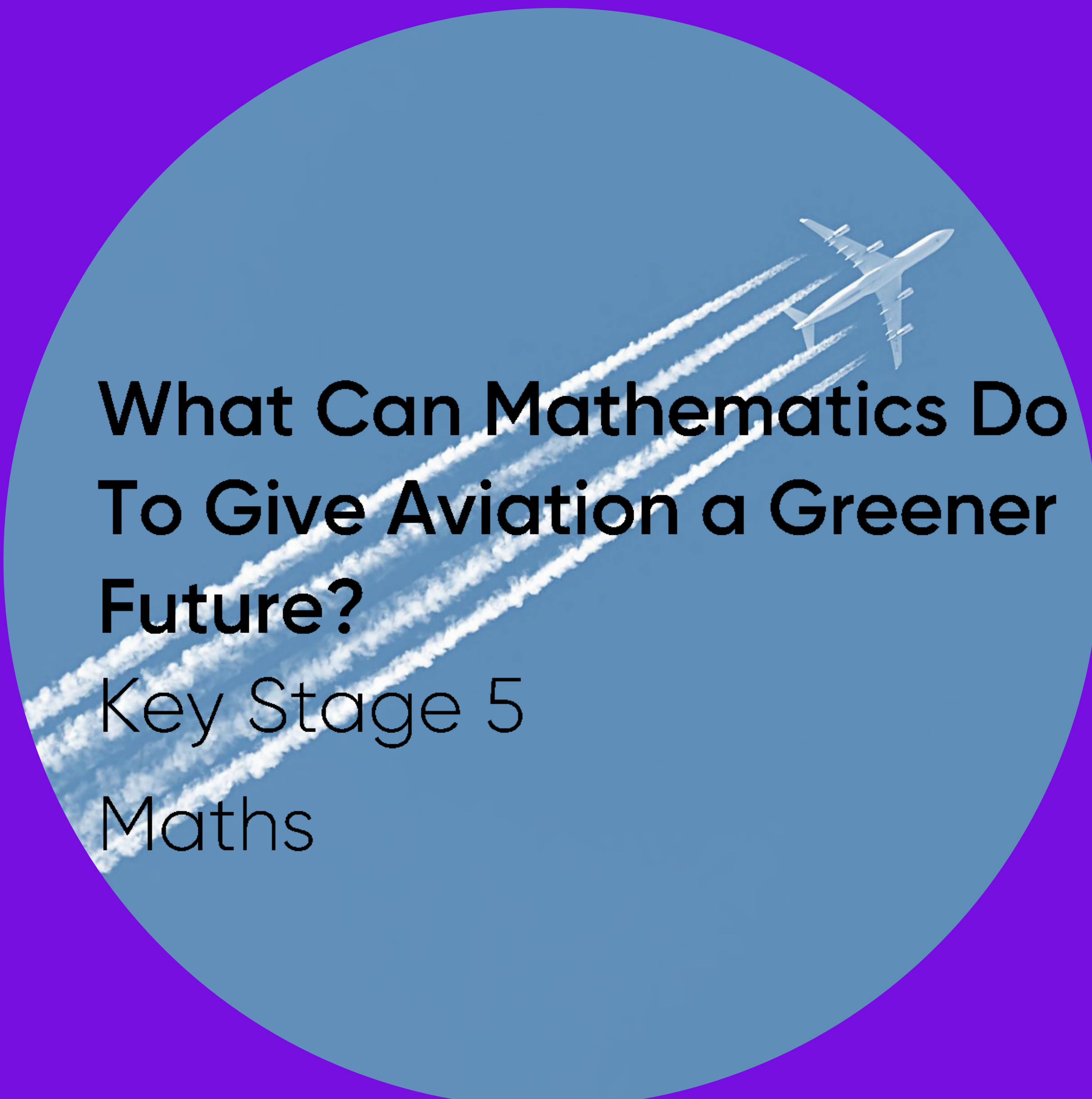
Research
Based
Curricula



What Can Mathematics Do To Give Aviation a Greener Future?

Key Stage 5

Maths

2020

BROUGHT TO YOU BY RBC ACCESS ED

Contents

For Students
Getting Started

RBC means Research-Based Curriculum,. Each RBC coursebook is written by a PhD student at a university about their cutting edge research.

Why complete an independent 'RBC' study pack?

RBC courses are challenge courses to sharpen your skills and resilience: finishing a RBC course is a major accomplishment to add to your academic CV. To get into the university, you must demonstrate that you are intellectually curious, and will make the most of the academic opportunities available to you. Completing a pack will allow you to gain invaluable experience to write about in your university application..

It allows you to:

✓ Build your subject experience to mention in your UCAS Personal Statement

✓ Sharpen your academic skills

✓ Experience what it's like to study beyond school and at university

✓ Better understand what you enjoy and don't

✓ Improve your overall subject understanding ahead of final exams

For Students
Getting Started

What's in this booklet?

Your RBC booklet is a pack of resources containing:

- ✓ More about how and why study this subject
- ✓ Six 'resources' each as a lesson with activities
- ✓ A final assignment to gauge learning
- ✓ Extra guidance throughout about the university skills you are building
- ✓ End notes on extra resources and where to find more information

Who should complete this pack?

Anyone interested in improving their academic skills or understanding what they should do at university. *This pack is especially great for anyone interested in studying* ***Maths*****,** and the interesting ways that it can be applied in real life.

Even if you are unsure of where your interest in these subjects can take you, by completing this pack you will have a clearer idea of the variety of subjects that link to one another.

If you have any questions while you are using the resources in this pack, you can contact your teacher or email us directly at schools@access-ed.ngo.

Good luck with your journey to higher education!

For Students
University Skills

University Ready!

To complete this resource, you will have to demonstrate impressive academic skills. When universities are looking for new students, they will want young people who can study independently and go above and beyond the curriculum. All of these skills that you will see here will demonstrate your abilities as a university student – while you're still at school!

Every time you have to look something up, or write up a reference you are showing that you can work independently.

Every time that you complete a challenging problem or write an answer to a difficult question, you might demonstrate your ability to think logically or build an argument.

Every time that you evaluate the sources or data that you are presented with, you are showing that you can "dive deep" into an unfamiliar topic and learn from it!

Skills you will build for university:

independent research	your ability to work on your own and find answers online or in other books
creativity	your ability to create something original and express your ideas
problem solving	your ability to apply what you know to new problems
building an argument	your ability to logically express yourself
providing evidence	your ability to refer to sources that back up your opinions/ ideas
academic referencing	your ability to refer to what others have said in your answer, and credit them for their ideas
Deep dive	your ability to go above and beyond the school curriculum to new areas of knowledge
source analysis	your ability to evaluate sources (e.g. for bias, origin, purpose)
Data interpretation	your ability to discuss the implications of what the numbers show
Active reading	your ability to engage with what you are reading by highlighting and annotating

Where can this subject take me?

Pathways

Studying Biology or Psychology can open the doors to many degrees and careers. It intersects with microbiology, chemistry, physiology, and sociology. Whatever interests you is likely to relate to biology in some way. See a snapshot of where studying Biology and Psychology can take you.

'Transferrable skills' from Maths to a career:

- Attention to detail
- Good at investigating, analysing and interpreting data, finding patterns and drawing conclusions
- information technology
- approaching problems in an analytical way
- dealing with abstract concepts
- Good at analysing large quantities of data
- Logical thinking

What are some are the 'interdisciplinary' subjects in this course?

Interdisciplinary is a term you will hear used by higher education institutions. It's also how many professionals and academics in the real-world operate: they use multiple subjects, or disciplines, to achieve their work.

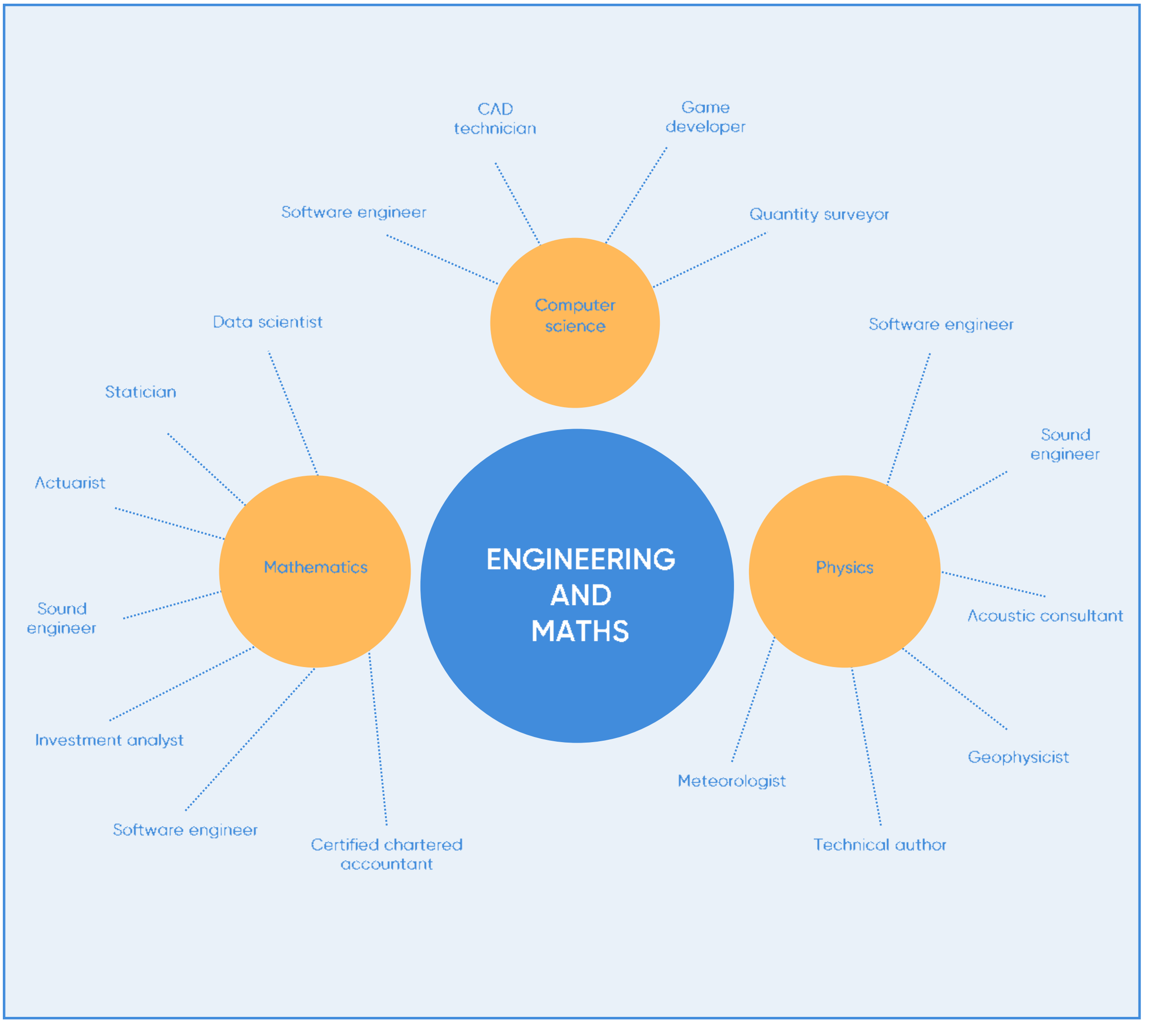
By thinking about which subjects you like, alongside maths, it can help you choose a career pathway later.

Read more about subject selection and careers pathways:

https://targetjobs.co.uk
https://www.prospects.ac.uk
https://thinkuni.org/

Subject map: Engineering and Maths

Students with an engineering degree are often seen with a background in mathematics and physics as this is essential in the principles of projects that they carry out. Students with a mathematics degree do into jobs in finance as well as IT services. Students studying mathematics and physics can also go into academia for further study in the subject.



Find our about Science-related careers here:
PROSPECTS: https://www.prospects.ac.uk
TARGET JOBS: https://targetjobs.co.uk

For Teachers RBC Guide

Learner aims

The Research-Based Curriculum aims to support student attainment and university progression by providing classroom resources about cutting-edge research at local universities. The resources are designed to:

- ✓ promote intellectual curiosity through exposure to academic research
- ✓ stretch and challenge students to think deeply about content that may be beyond the confines of the curriculum
- ✓ develop core academic skills, including critical thinking, metacognition, and written and verbal communication
- ✓ inform students about how subjects are studied at university, and provide information, advice and guidance on pursuing subjects at undergraduate level

Content

The programme represents a unique collaboration between universities and schools. Trained by AccessEd, PhD Researchers use their subject expertise to create rich resources that help bring new discoveries and debates to students.

The Research-Based Curriculum offers ten modules suitable for either KS4 or KS5 study. The modules span a range of disciplines, including EBacc and A-level subjects, as well as degree subjects like biochemistry. Each module includes six hours of teaching content, supported by student packs, teacher notes and slides. All modules are available online and free of charge for teachers at select schools.

Using the RBC pack

These resources are designed to be used flexibly by teachers. The resources can be completed by students individually or in groups, in or out of the classroom.

For Teachers
Using the RBC packs

Extra-Curricular Subject Enrichment Clubs

Here are five examples of delivery options:

The resources can be completed in small groups (4-8 pupils) across a series of weekly lunch clubs or after-school clubs. Groups can reflect on their learning by presenting a talk or poster on the subject matter at the end of the course.

University Access Workshops

The resources can be used by students to explore subjects that they are interested in studying at university. This can inform their decision making with regards to university degree courses, and allow students to write more effective personal statements by including reflections on the Research-Based Curriculum.

Research Challenge

The resources can be used to ignite curiosity in new topics and encourage independent research. Schools could hold a research challenge across a class or year group to submit a piece of work based on the resources. Pupils could submit individually or in small groups, with a final celebration event.

Summer Project

Resource packs can function as 'transition' projects over the summer, serving as an introduction to the next level of study between KS3 and KS4, or KS4 and KS5. Students could present their reflections on the experience in a journal.

Why offer these?

The Research-Based Curricula programme builds on the University Learning in Schools programme (ULiS), which was successfully delivered and evaluated through the London Schools Excellence Fund in 2015. The project was designed in a collaboration between Achievement for All and The Brilliant Club, the latter of which is the sister organisation of AccessEd. ULiS resulted in the design and dissemination of 15 schemes of work based on PhD research for teachers and pupils at Key Stage 3. The project was evaluated by LKMCo. Overall, pupils made higher than expected progress and felt more engaged with the subject content. The full evaluation can be found here: ULiS Evaluation.

Questions

For more information contact hello@access-ed.ngo

Introduction to the Topic

Studying Mathematics at University

Forget about getting to the right answer! Mathematics is so much more than that at university. For a start how do you know that your so-called right answers are actually valid? There is no answer page to check now. The real beauty of Mathematics lies in spotting some really interesting patterns and then trying to pin these down to a general formula. If you can prove that your theory works, it might be your theorem that students up and down the country have to learn for their exams! When you first start studying Mathematics at university there will be a lot of areas to consolidate, but the material will quickly move from familiar work to ideas that really dig down into the fundamental essence of what Mathematics really is. Don't expect everything to have an instant application either. Much of the theory you will learn can be used in all sorts of real-world problems, but ideas that don't seem that useful now could provide the backbone of future research. For example, 18th century mathematicians used to play with prime numbers for fun; now they are essential in maintaining internet security. Within a Mathematics degree there are a whole range of different options that you will be able to choose from. It is worth checking the prospectus of each university course very carefully to make sure that the areas you particularly enjoy are all there. You might like the n dimensional canvas that is provided by fractal geometry, where n doesn't even have to be a whole number. You could try the more conventional area of linear algebra that leads into areas of coding, including cryptography. Perhaps you enjoyed Mechanics or Statistics in your A level course and now want to try some Statistical Mechanics!

Introduction to the Topic

Studying Mathematics at University

The problem is that Mathematics is just too vast for everything to fit into one degree! That is where a Masters degree can help out, with the opportunity to move further into a particular area of the subject, or just try some topics you did not have the chance to look at in your first degree. From there it is possible to move on to a PhD, which will tend to specialise in one or two keys areas. The methods you use will be innovative, so you will be carving out your own little chunk of mathematical theory, making links and gaining results that no-one has ever seen before!

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Introduction to Research

The topics within this pack will include:

Working with vectors

Gradients and partial differentiation

Numerical gradients and interpolation

My PhD continues from a year long Masters of Research course into the Mathematics of Planet Earth. The idea is that everyone on my course is busy finding a new way to help to solve some of the problems of climate change, from limiting its causes to measuring what is going on around us. My contribution to this important work revolves around making aviation more sustainable. This is a crucial area, as about one twentieth of all anthropogenic climate change (ie climate effects that are directly attributable to man's influence) can be traced back to the aviation industry. Every year over 800 million tonnes of carbon dioxide are emitted into the atmosphere by commercial flights. One argument would be to ground all of the planes, but with 4.6 billion passengers relying on catching flights annually and over six trillion dollars worth of freight flown around the world every year, we can't just ban flying! There have been many different innovations put forward to solve this problem, or at least improve the situation. Some involve creating new fuels, others rely on new aeroplane design. However, these tactics are long term, high cost approaches. The only real method to make airlines adopt new initiatives is to help them to save money, whilst they are improving the environment. That is where route planning comes into its own. If we can create new routes that allow aircraft to fly more efficiently, by taking into account the wind fields they are flying through, then less fuel will be burned and thus less pollution will result. In the first stage of my studies, I applied calculus of variations and optimal control theory to obtain the minimum time for a journey between London Heathrow Airport and John F Kennedy Airport in New York.

Introduction to Research

This involved working out how an aeroplane's position in space changed with time and then choosing the correct heading angle to allow the aircraft to intercept the most useful wind fields across the North Atlantic. In this way a dynamical system could be established and then solved numerically. Once minimum time trajectories had been plotted at each air speed, these times could be combined with equations giving fuel burn rate. In this way the most fuel efficient route for each day's wind patterns could be found. The next step will be to vary the air speed within a flight path and look at changing altitude to intercept more advantageous weather conditions too.

Meet the PhD Researcher Cathie Wells

My journey to becoming a PhD student has not been conventional. Having studied Mathematics, Further Mathematics, Physics and French at A level, I then decided to take a degree combining Mathematics and French Language and Literature. This was a bit of a departure from my original choice of Engineering, but having realised one term in that my natural skill set did not include creating structures from angle iron, I decided to try something different. Mathematics and French may seem an unusual combination, but I even managed to complete an interdisciplinary dissertation setting the ideas and work of Evariste Galois (well worth looking up if you thought all mathematicians led ordinary lives!) against the romantic era of literature and revolutionary politics into which he was born. I re-proved some of his ideas using modern mathematical methods and wrote the whole project in French. I was offered a PhD place at a French university to study Mathematics further after my BA. However, at that stage I felt that I needed to start earning some money, so instead I took a one year PGCE course in teaching Mathematics. My idea was to teach for a bit and then go back to university and finally complete a PhD. As things turned out, I ended up teaching for the next twenty-one years and thoroughly enjoyed sharing some of my favourite areas of Mathematics with my students. However, I knew there was still more that I needed to learn, so for the last four years of my teaching career, I also studied for a Masters degree in Mathematics with the Open University. After so long being the expert, going back to knowing.

Meet the PhD Researcher Cathie Wells

After so long being the expert, going back to knowing very little was tough, but I like to think it made me more sympathetic when my students had difficulties with theory or fitting in adequate revision! Once I'd finished my MSc, I decided to look around for the next opportunity and came across the course on which I am now enrolled. Last year I completed my Masters of Research and now I am embarking on my doctorate in Mathematics of Planet Earth. By the time I finish my PhD, I will still have time for 21 years of a whole new career, so this is almost like being back to where I left off, except that along the way I've managed to acquire a husband, three children and two cats!

A-Level Subjects	Mathematics, Further Mathematics, Physics and French
Undergraduate	Mathematics and French Language and Literature
Postgraduate	Master in Mathematics, PhD Mathematics of Planet Earth

Glossary

Term	Definition
Air speed	speed an aeroplane would be going in still air
Analytic solution	given through exact mathematical solution, most notably involving calculus
Array	set of data stored in more than one dimension.
Critical points	stationary point for both variable directions of a surface
Data cleaning	removing or correcting any false data
Differential equation	equation which includes at least one term which is a derivative
Dynamical system	set of differential equations that are linked by context
Euler's Method	numerical method to solve differential equations
Ground speed	combination of air speed and the wind speed of a plane at any point.
Interpolation	method for finding data between the given data points
Meridional	in the North/South direction.
Numerical solution	given through an approximation using a numerical method.
Partial derivatives	differentiation with respect to one variable of a function of more variables.
Saddle point	point with a relative maximum in one direction and a minimum in the other.
Rate of change	way in which one variable changes with respect to another.
Relative velocity	velocity of one body as seen from another moving body.
Taylor's Series	summation of derivatives and polynomial terms to approximate a function.

Glossary

Term	Definition
Trajectory	path describing motion
Wind field	set of all vectors describing wind direction and magnitude across a region.
Zonal	in the West/East direction.

Resource One Overview

Topic

Working with vectors

A-level Modules

Use vectors in two dimensions and in three dimensions.

Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.

Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.

Objectives

By the end of this resource, you will be able to:

- ✓ Use vectors in describing relative speeds and positions.
- ✓ Understand how trigonometry and geometry is necessary in solving vector problems.
- ✓ Set up a dynamical system.
- ✓ Looking at the difference between a 2D flat system and a spherical system.

Instructions

1. read the source
2. answer the questions
3. explore the further reading
4. move on to the next resource

Resource One
Data Source

Section A

Relative Motion on the Road

If you are travelling at 110 km/h along a dual carriageway and someone overtakes you at 130 km/h, how fast do they seem to be going to you? What if you were travelling in the opposite direction from someone driving at this speed? We talk about speed, but actually, mathematically we mean velocity, as the cars have both speed and direction. How is this represented in the example below?

Problem Solving

<u>Scenario 1:</u>

 V_{you} 110 km/h

 V_{them} 130 km/h

${}_{them}V_{you} = V_{them} - V_{you} = 130-110=20$ km/h

<u>Scenario 2:</u>

 V_{you} 110 km/h

V_{them} -130 km/h

${}_{them}V_{you} = V_{them} - V_{you} = -130-110=-240$ km/h

It is reasonably easy to see that the overtaking car seems to go at 20 km/h compared with you, but the car in the opposite direction looks like it is going at a blistering 240 km/h. In effect you are ignoring the road and making your own movement the value counted as "still".

Section B

Relative Motion in Two Dimensions

This effect becomes a little more complicated once we leave straight roads and take to the water or the air.

If someone wants to row directly across a river, the current will have to be taken into account. The velocity of the boat with respect to the river (boatVriver) tells you where the boat would go if the river was staying perfectly still.

Resource One
Data Source

Active Learning

Try the example below. You'll need to use angles here and your final answer can be expressed as a bearing or in terms of compass points.

The boat is capable of travelling at 10 m/s in still water.
The river is flowing at 6 m/s.
What course must the boat steer to reach the other bank directly opposite to its starting position?
If the banks are 50 m apart, how long will the journey take?

Working:

NB: Don't forget that the speed of the boat in still water is its relative speed, not the speed it actually crosses at.

${}_{boat}V_{river} = V_{boat} - V_{river} \Longrightarrow {}_{boat}V_{river} + V_{river} = V_{boat}$

θ

6 m/s

10 m/s

$\sin\theta = 6/10 \Longrightarrow \theta = 36.9^{o}$ to 3sf
So the boat must be directed on a bearing of 127^{o} or at S53.1 ^{o}E

In still water the boat would go at 10 m/s, so using Pythagoras' theorem it really goes at 8 m/s. This gives a time for the 50 m crossing of 6.25 s.

Resource One Activities

Activities

Active Learning

Exercise 1 : Crossing the river

Now try the example in the resource again, but this time you need to cross as quickly as possible. What does this mean about the direction of your velocity arrows? Try a few different routes and check your time. How can you minimise this value?

Exercise 2 : Which path?

If you know what a journey looks like from two perspectives, this is often enough to pinpoint what the actual path taken and speed of travel are. If a delivery driver going due East at 30 m/s thinks that the plane she can see is going S50°W, but to a train driver traveling due North at 80 m/s it appears to be flying S30°W, on what bearing is the plane flying and how fast is it going?

Hints:

Split the problem into steps:

1. Plot the paths of the delivery driver and the train driver, going away from the same point, X. Remember to draw the length of the vector arrows to represent the speed. (If you work to scale, you can measure your answers from the diagram, but try the problem using trigonometry as well and check your answers against each other.)
2. At the end of each arrow, add on the arrow showing where the plane seems to go according to that person. Extend these vectors until they cross, call this point P.

Resource One Activities

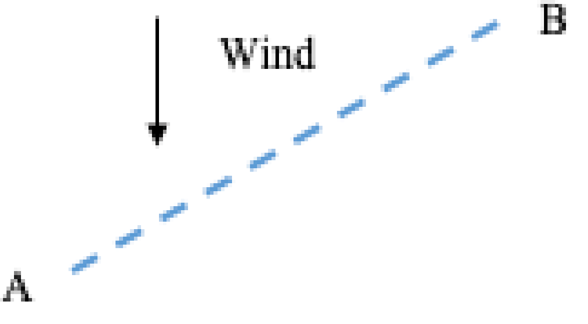
Activities

3. The line joining the X to P is the vector of the plane's movement.
4. Use the cosine and sine rules on a variety of triangles to get the length of the plane's velocity vector and the bearing of the plane. (NB: This is not easy…try joining the ends of the two drivers' paths to form a large triangle to get you started, before you look at the separate triangles made from each driver's view. Look out for an instance of the ambiguous sine rule too; a decent diagram will help you to spot this.)

Exercise 3: Windy weather

Now imagine a plane trying to fly from Airport A to Airport B, where B is on a bearing of 060o from A and 300 km away. The wind is blowing from the North at 50 m/s.

Where should the pilot direct the plane? How long will the flight take if the air speed of the plane is kept to a steady 200 m/s?



NB: We are looking at this as a "flat" problem, so imagine you are seeing the plan view of the route to the airport. Here you imagine that the airports and the plane's flightpath are all at the same vertical level.

Resource One
Data Source

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Section C

Expressing aircraft motion in terms of vector components

Problem Solving

In actual fact a transatlantic flight is about 5 500 km. This means that the wind varies quite a lot across this distance. Wind data tends to be given in an array of zonal and meridional components for each point across a grid of 2.5 degree squares. Of course, these are not really square at all due to the curvature of the Earth's surface. Look at a globe. One degree of latitude will always be represented by the same distance in the meridional direction on the globe. So the spherical grid has a fixed meridional height. However, if you measure the zonal grid lengths, these change, despite representing the same number of degrees of longitude, depending on how far you are from the poles.

To start with, let's go back to the flat system:

At the point (x,y) we will call the zonal wind $u(x,y)$ and the meridional wind $v(x,y)$.

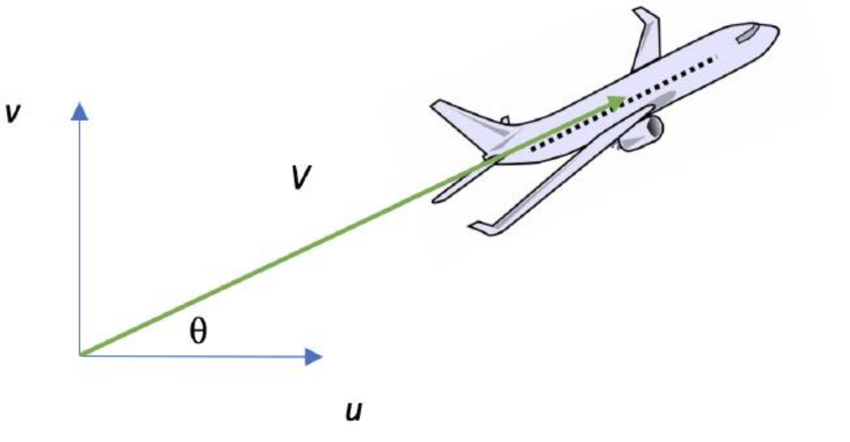
The air speed is set to W and represents what speed the plane would be flying at in still air. Its heading angle, which is always measured anti clockwise from due East is represented by θ. Ground speed is the speed the plane actually moves at once wind has been incorporated.

Resource One Activities

Activities

Exercise 4: Taking off

Can you find expressions for the zonal and meridional ground speeds? These will be dx/dt and dy/dt respectively.



Exercise 5:

If we now work with the spherical system, how do these expressions change?

Hint: It is easier to use positions in terms of latitude and longitude in degrees for this part, rather than thinking of x and y as distances.

These sets of differential equations are called dynamical systems, but this one is not yet complete, as there is another variable that will be changing, the heading angle. In order to find this last equation, you will need to know about partial differentiation.

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Resource One
Further Reading

Explore

Vectors:

- https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/4-5-relative-motion-in-one-and-two-dimensions/
- https://www.wired.com/story/in-physics-crossing-a-river-is-just-like-landing-a-plane/

Spherical geometry:

- https://www.movable-type.co.uk/scripts/latlong.html

Aeroplane trajectories in 3D:

- https://journals.ametsoc.org/doi/pdf/10.1175/1520-0469%281949%29006%3C0150%3AOTUOWI%3E2.0.CO%3B2

Independent Research

Resource Two Overview

Topic Gradients and partial differentiation

A-level Modules

Understand and use the derivative of f (x) as the gradient of the tangent to the graph of $y = f\,x$ at a general point (x, y)

Differentiate x^n, for rational values of n , and related constant multiples, sums and differences.

Differentiate $\sin kx$, $\cos kx$, $\tan kx$ and related sums, differences and constant multiples.

Apply differentiation to find gradients and stationary points.

Understand and use the second derivative.

Integrate x^n (excluding $n = -1$), and related sums, differences and constant multiples.

Integrate $\sin kx$, $\cos kx$ and related sums, differences and constant multiples.

Objectives

By the end of this resource, you will be able to:

- ✓ Extend differentiation into partial differentiation.
- ✓ Understand how to find maxima, minima and saddle points of surfaces.
- ✓ Work with the Hamiltonian to describe a dynamical system.
- ✓ Complete the dynamical system from Resource 1.

Instructions

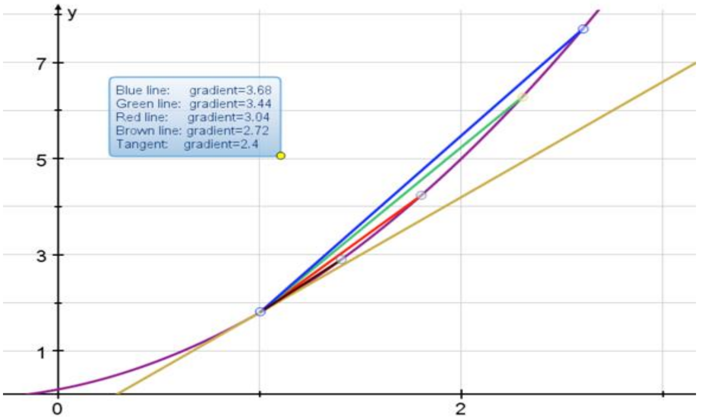
1. read the source
2. answer the questions
3. explore the further reading
4. move on to the next resource

Resource Two
Data Source

Section A

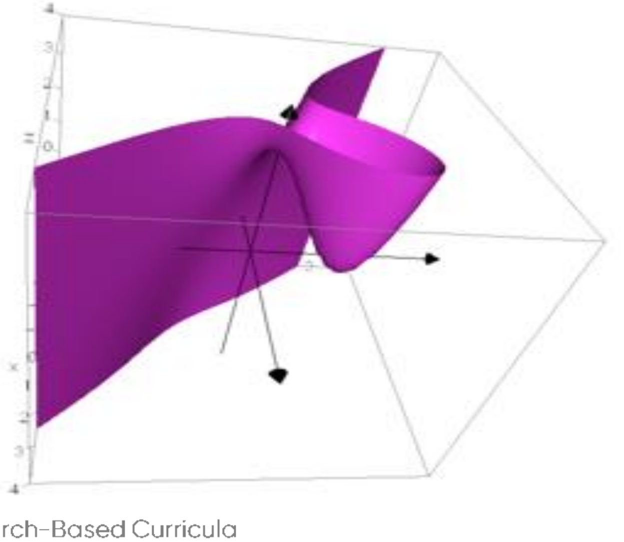
Differentiation of more than one variable

You should be aware that the gradient of a function can be found by differentiating. This is a way of finding the gradient of a chord of your curve and then sliding one end of your chord to the other until you have the gradient at a particular point.



Active Reading

This is fine in two dimensions on a flat grid, but what if your curve was now a surface? This is still defined as a 2D shape, it just happens to be bent and twisted in 3D, like taking a sheet of A4 paper and scrunching it up. We can't talk about turning points in quite the same way any more. Before it was a case of finding the rate of change of the function with respect to one variable, . Now we have two different variables, each contributing to the function and in some cases we will have combined terms involving both. This means that we will need to differentiate in two directions and compare the results we obtain.



Resource Two
Data Source

This is called partial differentiation, as we will only differentiate with respect to one variable at a time.

Here are some examples to show you how this works:

$$z = x^2 + 3y^2 - 2x + 12y - 9$$

$$\frac{\partial z}{\partial x} = 2x - 2$$

Notice that we assume y is constant as it has no x component.

$$\frac{\partial z}{\partial y} = 6y + 12$$

This time take x as constant. Having two first partial derivatives, means that we can now have four second partial derivatives.

$$\frac{\partial^2 z}{\partial x^2} = 2 \qquad \frac{\partial^2 z}{\partial y^2} = 6 \qquad \frac{\partial^2 z}{\partial x \partial y} = 0 \qquad \frac{\partial^2 z}{\partial y \partial x} = 0$$

Activities

Active Learning

$$z = x^3 - 2y^2 + 3xy - 2x^2y$$

Show that:

$$\frac{\partial z}{\partial x} = 3x^2 + 3y - 4xy$$

$$\frac{\partial z}{\partial y} = -4y + 3x - 2x^2$$

Hint:

When a mixed term is used then one variable is regarded as a constant coefficient

Now there will be a more interesting set of second derivatives. Try finding these. Are there any patterns? (Hint: try researching Clairaut's Theorem.)

Resource Two
Data Source

Section B

Classifying critical points

Critical points are located wherever $\partial z/\partial x$ and $\partial z/\partial y$ are both equal to zero.

So in the first example there is only one critical point: (1,-2,-22).

(Substitute x and y back into the original function to find z.)

To classify this point we need to find $D = (\partial^2 z)/(\partial x^2) \times (\partial^2 z)/(\partial y^2) - ((\partial^2 z)/\partial x \partial y)^2$ at the critical point.

If $D>0$ and $(\partial^2 z)/(\partial x^2)<0$ then the critical point is a relative maximum. (Like a hill summit.)

If $D>0$ and $(\partial^2 z)/(\partial x^2)>0$ then the critical point is a relative minimum. (Like a sink hole.)

If $D<0$ then the critical point is a saddle point.

(This is a maximum in one direction and a minimum in the other, think Pringle!)

If $D=0$ then more clarification is needed.

In our first example we can classify (1,-2,-22) as a relative minimum.

Activities

Find and classify all of the critical points of:

i) $z = 5x^3 + 2y^2 - 60xy - 3$

ii) $z = x^3 - 8y^3 - 2x^2y + 4xy^2 - 4x + 8y$

Resource Two Data Source

Section C

Classifying critical points

Active Learning

When we have a set of differential equations to solve, we can use something called the Hamiltonian of the dynamical system. This is often used where a spatial position changes with time. If a flat 2D system is Hamiltonian, then:

$dx/dt = \partial H/\partial y$ and $dy/dt = -\partial H/\partial x$ and we can find H by integrating directly.

Normally you would need to include a constant of integration in your answer, but here, if you integrate with respect to y, you need to include some arbitrary function in x, called $f(x)$ and if you integrate with respect to x, you need to include some arbitrary function in y, called $g(y)$. These functions could be any function in these variables, including a constant, but they cannot be functions of both x and y.

Example 1: Given the following Hamiltonian system, find H:

$$\frac{dx}{dt} = x^2 - x\cos y + 3y^2$$

$$\frac{dy}{dt} = -2xy + \sin y + 3$$

Rewrite these as:

$$\frac{\partial H}{\partial y} = x^2 - x\cos y + 3y^2$$

$$\frac{\partial H}{\partial x} = -(-2xy + \sin y + 3)$$

Integrate the first line with respect to y:

$$H = x^2y - x\sin y + y^3 + f(x)$$

Integrate the next line with respect to x:

$$H = -(-x^2y + x\sin y + 3x) + g(y) = x^2y - x\sin y - 3x + g(y)$$

This gives our Hamiltonian as:

$$H = x^2y - x\sin y + y^3 - 3x$$

Resource Two
Data Source

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Activities

Exercise 3:

Which of the following systems are Hamiltonian? Work out H for each of these:

i)

$$\frac{dx}{dt} = 4x^2y + 3xy^2 - 8x$$

$$\frac{dy}{dt} = 8y - y^3 - 4xy^2$$

ii)

$$\frac{dx}{dt} = 3xy^2 + x\cos y - 2x^2y$$

$$\frac{dy}{dt} = 2xy^2 - y^3 - \sin y + y$$

iii)

$$\frac{dx}{dt} = 3x^2 + y\sin x - 4xy^2$$

$$\frac{dy}{dt} = -\frac{y^2}{2}\cos x + \frac{4y^3}{3} - 6xy - \tan x$$

So why would we want to know what the Hamiltonian is?

You'll notice that H does not vary explicitly with time, only with spatial variables x and y. So by choosing a constant value for H we can start to see how y varies with x. When plotted this gives something called a "phase path". Choosing different H values allows us to plot further phase paths, until we have a pretty good idea how x and y are related at different times. To find out more about this look at the further reading resources.

Resource Two
Data Source

Section D

Deriving a dynamical system for aeroplane flight

Active Learning

Now back to aviation. We had a dynamical system for the 2D model for a plane moving across the North Atlantic in Resource 1:

$$\frac{dx}{dt} = W\cos\theta + u(x,y)$$

$$\frac{dy}{dt} = W\sin\theta + v(x,y)$$

Remember that u and v are the zonal and meridional wind speeds at any point (x,y), that can be found given our array of wind data. W is the air speed of the plane and θ is the heading angle.

In Optimal Control Theory we also need a cost functional which stems from what we are trying to optimise. Here we need to minimise time, so the cost functional is just the negative of the time the whole flight takes. Within the Hamiltonian expression it is represented by the extra constant term +1. We now include two adjoint variables which ensure that we have a Hamiltonian system, these are called〚λ〛_x and λ_y. By finding what these are, we will be able to derive the third line of the dynamical system, the one which describes how θ changes with time. So the Hamiltonian is given as:

$$H = \lambda_x(W\cos\theta + u(x,y)) + \lambda_y(W\sin\theta + v(x,y)) + 1$$

Resource Two
Data Source

BROUGHT TO YOU BY RBC ACCESS ED

Activities

Exercise 4:

i) Find the partial derivatives of H with respect to x, y and θ.

ii) Set $\frac{\partial\lambda_x}{dt} = -\frac{\partial H}{\partial x}$ and $\frac{\partial\lambda_y}{dt} = -\frac{\partial H}{\partial y}$.

iii) By setting $\partial H/\partial\theta$ to zero, find the relationship between λ_x and λ_y.

iv) As H is not explicitly a function of time, choose to set it to zero.

v) Rearrange your answer to iv) to make λ_x the subject of the formula.

Vii) Substitute your answers to v) and vi) into your $(\partial\lambda_x)/dt = -\partial H/\partial x$ and $(\partial\lambda_y)/dt = -\partial H/\partial y$ expressions.

viii) Now find $d\theta/(\partial\lambda_x)$.

ix)Using the Chain Rule, find $d\theta/dt$ as the product $d\theta/dt = d\theta/\partial\lambda_x \times \partial\lambda_x/dt$.

x) Substitute in your expressions for λ_x and λ_y to give the final $d\theta/dt$ expression:

$$d\theta/dt = sin^2\,\theta\ \partial v/\partial x - cos^2\,\theta\ \partial u/\partial y + (\partial u/\partial x - \partial v/\partial y)\ sin\theta\ cos\theta$$

We now have the full set of differential equations to solve to give an answer for our time minimisation problem.

Resource Two
Further Reading

Explore

Now try looking at:

Classifying critical points of a surface:

- http://tutorial.math.lamar.edu/Classes/CalcIII/RelativeExtrema.aspx

Clairaut's Theorem:

- https://www.movable-type.co.uk/scripts/latlong.html

Dynamical Systems:

- https://www.youtube.com/watch?v=ZAVP-aFvCPs

Independent Research

Aeroplane trajectories in 2D using Optimal Control Theory:

- Applied Optimal Control by Arthur E. Bryson, Jr. (one to borrow from the library)

Resource Three Overview

Topic

Numerical gradients and interpolation

A-level module

Use numerical methods to solve problems in context.

Solve equations approximately using simple iterative methods.

Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy.

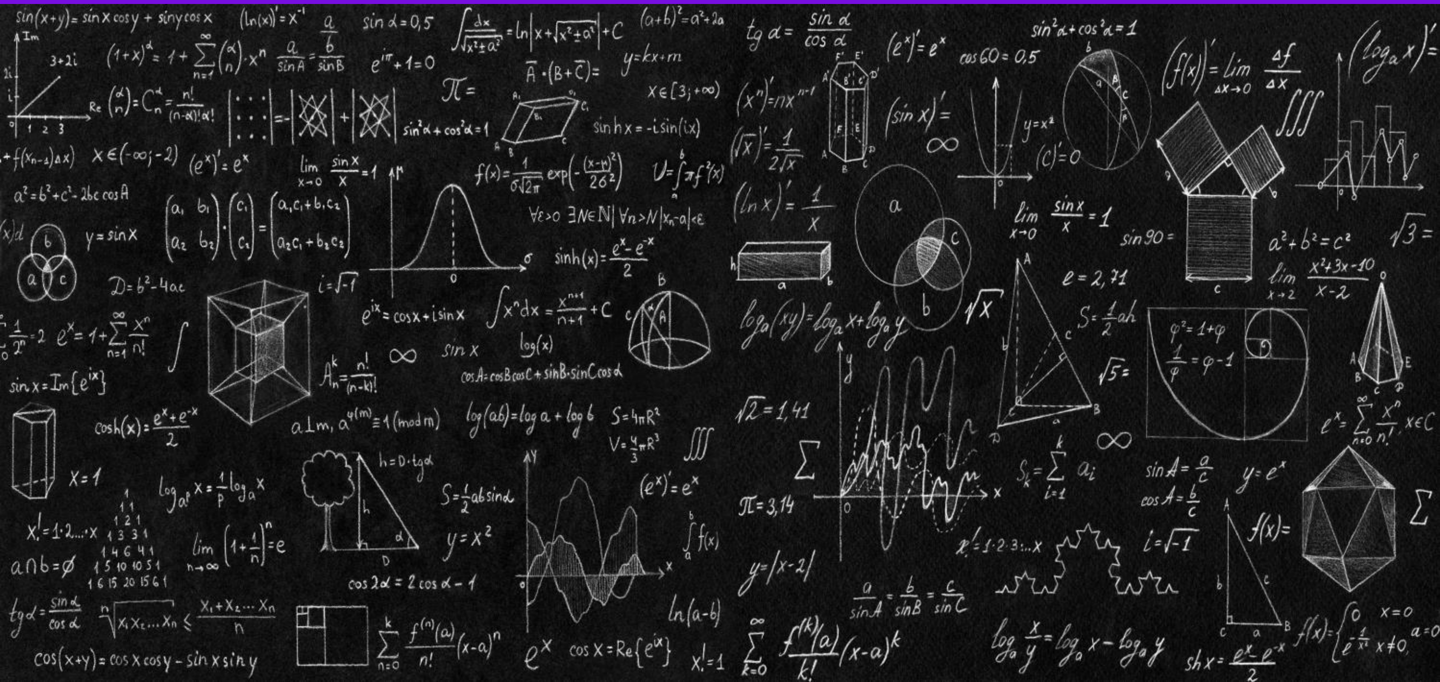
Objectives

After completing this resource, you should be able to:

- ✓ Learn to work with an array of data.
- ✓ Understand how gridded values can be interpolated in different ways.
- ✓ Find numerical gradients from gridded data.
- ✓ Apply this to wind fields along flight paths.

Instructions

1. read the source
2. answer the questions
3. explore the further reading
4. move on to the next resource



Resource Three Data Source

Section A

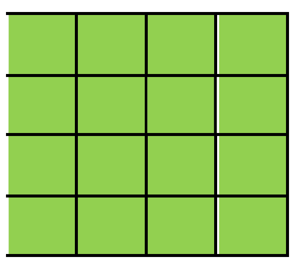
Interpolation methods

Often you will not be given a nice easy formula to use in solving a problem, but instead be faced with a large array of actual data. This is great, in that you have all of the answers already, but does limit how much data is available. For example if you placed a grid of 25 rain gauges in your garden for a month, you would know exactly how much rain had fallen in those places during that month, but if you now want to know if you could plant something at a point that is not directly in the vicinity of one of these and don't have time to move a gauge, you will need to try to work out the answer approximately from your current data.

Data Interpretation

There are several ways to do this. In this array are the data about the rain in your garden.

Each cell represents the answer for its grid position. The first position (1,1) of the array corresponds to the grid position in the top left corner of the garden.



The garden: split into 1m by 1m squares.

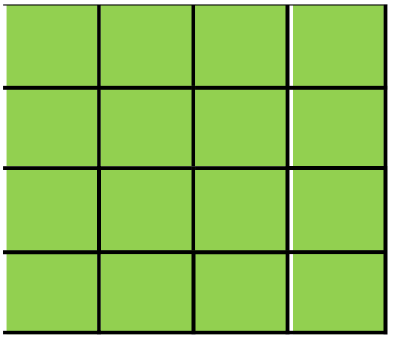
$$\begin{bmatrix} 8 & 9 & 4 & 3 & 2 \\ 7 & 5 & 6 & 2 & 2 \\ 9 & 6 & 4 & 3 & 8 \\ 4 & 5 & 3 & 4 & 6 \\ 4 & 5 & 5 & 3 & 4 \end{bmatrix}$$

The data: water depth in cm.

The amount of water collected by each gauge will depend on how sheltered an area is or whether water running off an obstacle might add to the total.

The blue, red and purple values correspond to the same colour crosses on the garden map.

Resource Three
Data Source



To work out an approximation for the rainfall where my star marker is placed we'll need to use the data around the point we're interested in. There are several ways to do this, but we'll look at three:

a) Nearest

b) Linear interpolation

c) Cubic interpolation

a) Nearest:

This just finds the gridded value nearest the new point. This is a useful way to interpolate if you need to use very little processing power. However, it does result in discontinuous data. If you are midway between the data values you will need to have a strategy for which value to take as nearest. Computer languages that have pre-programmed interpolation routines tend to choose the bottom right grid value, so here it would be 3 cm.

Resource Three
Data Source

Activities

Exercise 1:

i. Find the rainfall in a position (3.2, 5) using the nearest interpolation method.

ii. Find the rainfall in a position (2, 3.5) using the nearest interpolation method.

iii. Find the rainfall in a position (1.8, 4.2) using the nearest interpolation method.

iv. Write out the new array found by using the nearest interpolation method to split the top left original grid square into 9 squares.

v. Write out the new array found by using the nearest interpolation method to split the bottom right original grid square into 4 squares.

Section A

Interpolation methods

b) Linear interpolation:

This takes the value as the average of the four corner values if the point is in the middle of a grid space. The value shown would be 3.75 cm. If the point is on a grid row, but off the grid column then linear interpolation takes the correct number of steps in that direction:

(1.2,2) would put a plant where the red circle is. This is in line vertically with the 7 value in the array, but is 0.2 steps towards the 5 horizontally. Interpolation gives: $7+0.2\times(5-7)=6.6$ cm.

(4.1,4.8) would put a plant where the purple square is. This time we step 0.1 forwards on both the 4th and 5th rows, but then once we have these new positions, we move 0.8 down from the 4th row value towards the fifth row value.

Here we concentrate only on the smaller grid $\begin{bmatrix}4 & 6\\3 & 4\end{bmatrix}$ from the intersection of the 4th and 5th row and column.

Resource Three
Data Source

Moving 0.1 horizontally from both the 4 and the 3 on the left gives:

$4+0.1\times(6-4)=4.2$

$3+0.1\times(4-3)=3.1.$

Now we move vertically 0.8 down from the 4.2 to the 3.1:

$4.2+0.8\times(3.1-4.2)=3.32$ cm.

This is far more complicated than using nearest value interpolation, but it does guarantee continuous data. However, if you need to have data that has a continuous derivative, then a higher order method is necessary.

Activities

Active Learning

Exercise 2:

i. Find the rainfall in a position (3.2, 5) using the linear interpolation method.

ii. Find the rainfall in a position (2, 4.5) using the linear interpolation method.

iii. Find the rainfall in a position (1.8, 4.2) using the linear interpolation method. Does it matter if you interpolate vertically first and then horizontally here?

iv. Write out the new array found by using the linear interpolation method to split the bottom right original grid square into 4 squares.

v. If you have time, you could try splitting individual squares into more parts, but you will probably want to use a spreadsheet or write a quick computer programme to help you with this.

Resource Three
Data Source

Section A

Interpolation methods

c) Cubic interpolation:

On a grid like the garden, it is actually a bicubic interpolation that is applied.

This again works from the grid points around a value, but it needs the 4 closest data points in each direction and gives a continuous first derivative. It is a very useful method, but not really practical in our example without a computer to do all of the legwork. However, to give you a flavour of the method in 1 dimension, imagine you just have a row of data:

$$[4 \quad 6 \quad 4 \quad 3]$$

The first value of 4 corresponds to position 1, 6 to position 2 and so on. We need to find the data in position 2.6.

We then use the position (p) and the value $f(p)$ in a standard cubic equation:

$$f(p) = a \times p^3 + b \times p^2 + c \times p + d$$

to give a set of simultaneous equations:

$$\begin{aligned} 4 &= a \times 1^3 + b \times 1^2 + c \times 1 + d \\ 6 &= a \times 2^3 + b \times 2^2 + c \times 2 + d \\ 4 &= a \times 3^3 + b \times 3^2 + c \times 3 + d \\ 3 &= a \times 4^3 + b \times 4^2 + c \times 4 + d \end{aligned}$$

Solving these equations simultaneously will give values for a, b, c and d. We can substitute 2.6 into our equation to get the required value.

Resource Three
Data Source

Activities

Active Learning

Exercise 3:

i. Find the coefficients a, b, c, d.

ii. Find the value in position 2.6 using the cubic interpolation method. Hint: Just check that you get the correct coefficient values first before you put p in as 2.6.

$$\left(a = \frac{5}{6}, b = -7, c = \frac{103}{6}, d = -7\right)$$

iii. Repeat the process to find the value at position 5.9 in the vector below.

$$[2 \quad 7 \quad 9 \quad 3.4 \quad 12 \quad 9 \quad 5 \quad 4 \quad 1]$$

Hint: remember to use the two closest values on each side.

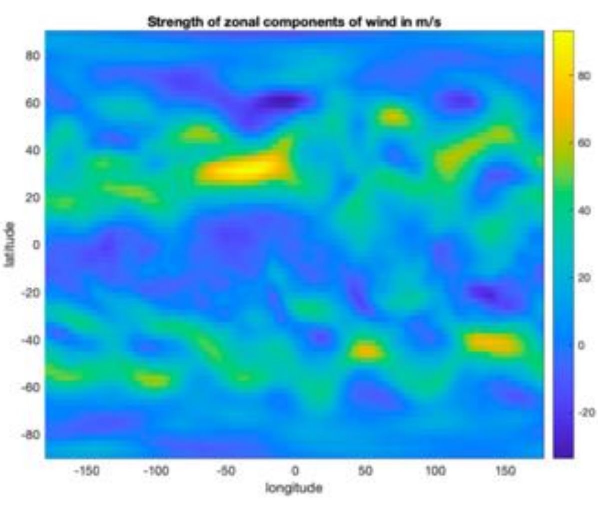
Section B

Interpolating a wind field

Thankfully any cubic interpolation will be completed using some form of computing code at university.

For the aircraft trajectories it is necessary to use interpolation in order to find the wind at any position of latitude and longitude across the North Atlantic, so that if in one time step the position does not lie on the grid, we can still be fairly certain that we can estimate the winds there. Luckily linear interpolation works fine here!

This chart shows the strength of just the zonal component of the wind across the world at 35 000 feet on 1st February, 2019.



Resource Three
Data Source

Activities

Exercise 4:

Here are two arrays of wind speeds for the start of a journey across the Northern Atlantic. The wind positions cover latitudes from 42.5 to 52.5 degrees North and longitudes 30 to 40 degrees West in 2.5 degree intervals. By using the list of positions given, work out the magnitude and direction of the wind at each point.

Zonal wind

$$\begin{bmatrix} 25.1 & 25.5 & 25.9 & 25.9 & 25.5 \\ 20.8 & 21.8 & 22.5 & 22.2 & 20.8 \\ 17.0 & 18.0 & 18.1 & 17.2 & 14.9 \\ 9.5 & 9.9 & 9.6 & 8.6 & 6.8 \\ 1.3 & 1.7 & 1.2 & 0.4 & -0.3 \end{bmatrix}$$

Meridional wind

$$\begin{bmatrix} 0.8 & 2.2 & 2.6 & 1.8 & 0.4 \\ 0 & 1.6 & 2.5 & 1.8 & 0.3 \\ -0.8 & 0.9 & 1.9 & 1.1 & 0.5 \\ -1.1 & 0.6 & 1.8 & 1.2 & 0.2 \\ -1.3 & 0.5 & 1.8 & 1.9 & 1.1 \end{bmatrix}$$

Positions:

(45N,32W)

(46N,35W)

(51.5N,39W)

Numbers in the same row are at the same latitude, those in the same column are at the same longitude. The first row gives 52.5 degrees North, the first column gives 40 degrees West.

Hint: Find zonal and meridional components of the wind using linear interpolation. Then find the magnitude of the wind using Pythagoras' Theorem and the direction using trigonometry.

Resource Three
Data Source

Section B

Numerical gradients

This style of estimation is also used to find numerical gradients. In the third line of the dynamical system discussed in Resource 2, we have the rate of change of zonal and meridional wind in the directions of longitude and latitude:

$$\frac{d\theta}{dt} = sin^2\theta\frac{\partial v}{\partial x} - cos^2\theta\frac{\partial u}{\partial y} + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) sin\theta cos\theta$$

To find the shaded values from our arrays of wind data, we will need to calculate numerical gradients and then interpolate to be able to give these at any point.

Activities

Exercise 5:

i. Numerical gradients for the zonal wind speeds above have been calculated for you. Can you work out how these have been obtained from the zonal wind array?

Zonal wind: gradient in x direction

$$\begin{bmatrix} 0.16 & 0.16 & 0.08 & -0.08 & -0.16 \\ 0.4 & 0.34 & 0.08 & -0.34 & -0.56 \\ 0.4 & 0.22 & -0.16 & -0.64 & -0.92 \\ 0.16 & 0.02 & -0.26 & -0.56 & -0.72 \\ 0.16 & -0.02 & -0.26 & -0.3 & -0.28 \end{bmatrix}$$

Zonal wind: gradient in y direction

$$\begin{bmatrix} -1.72 & -1.48 & -1.36 & -1.48 & -1.88 \\ -1.62 & -1.5 & -1.56 & -1.74 & -2.12 \\ -2.26 & -2.38 & -2.58 & -2.72 & -2.8 \\ -3.14 & -3.26 & -3.38 & -3.36 & -3.04 \\ -3.28 & -3.28 & -3.36 & -3.28 & -2.84 \end{bmatrix}$$

ii. Find values for $\partial u/\partial x$, $\partial u/\partial y$ at the three positions given in Exercise 4. Remember that you will need to use linear interpolation on the gradient arrays for the zonal winds.

iii. Repeat ii) for the meridional winds this time finding $\partial v/\partial x$, $\partial v/\partial y$. You will need to calculate the gradient array first. You might like to use a spreadsheet or a computer programme to speed this up.

Resource Three
Further Reading

Explore

Now try looking at:

Interpolation:

- https://www.khanacademy.org/partner-content/pixar/animate/parametric-curves/v/animation-5
- https://en.wikibooks.org/wiki/Introduction_to_Numerical_Methods/Interpolation
- https://www.youtube.com/watch?v=EtzlEA9MIwI

Wind data:

- https://www.esrl.noaa.gov/psd/data/composites/day/

Independent Research

Resource Four Overview

Topic	Solving Differential Equations
A-level Modules	Construct simple differential equations in pure mathematics and in context. Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions (Separation of variables may require factorisation involving a common factor). Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics. Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy.
Objectives	After completing this resource, you should be able to: ✓ Revise what differential equations mean. ✓ Understand that the majority of such equations are not solvable analytically. ✓ Introduce Taylor's series to allow the derivation of Euler's method. ✓ Learn to use Euler's method to solve differential equations. ✓ Apply this to the dynamical system derived in Resource 2, by using the wind fields and wind gradients from Resource 3.
Instructions	1. read the source 2. answer the questions 3. explore the further reading

Resource Four Data Source

Section A

Rates of change

Active Reading

Differential equations can be linked with almost anything in life. To test this, try running an internet search for any hobby or object and differential equations. You will almost always find an equation that is linked to your topic, from ice cream making to tae kwondo. The reason for this is that differential equations look at rates of change and in any activity, something is changing with respect to something else. Normally this will be something with respect to time. For example in ice cream making there are freezing rates to take into account, in tae kwondo there will be speed of punches or kicks. To remind you of some of this theory, you'll need to start by creating a couple of basic differential equations from rates of change.

Example:

A blob of ink spreads out on blotting paper to be a perfect circle. If the rate of change of the radius with time is 2 cm/s create a differential equation to describe the rate of change of the area of the circle with time. Using your answer, find the rate of change of the area when the radius is 4cm.

Answer:

$$\frac{dr}{dt} = 2$$

$$A = 2\pi r^2 \Longrightarrow \frac{dA}{dr} = 4\pi r$$

$$By\ the\ Chain\ Rule: \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \Longrightarrow \frac{dA}{dt} = 4\pi r \times 2 = 8\pi r$$

$$\text{When r=4 cm} \Longrightarrow \frac{dA}{dt} = 32\pi$$

Resource Four
Data Source

Activities

Exercise 1:

i. Water is leaking from the vertex of an inverted cone at the rate of 5 cm^3 per minute.

What is the rate of change of the depth of the water in the cone? If the cone is initially full of water and its radius and height are equal, what will the rate of change of depth be when the depth has halved?

Hint: Remember that dh/dV is just the reciprocal of dV/dh.

ii. An open cubical box of side q is filled with sand at a rate of $6q^2$ cm^3/s, but the sand is also leaking out of the bottom of the box at a rate of $4q$ cm^3/s. What is the rate at which the depth of sand in the box increases? As q becomes larger the rate of depth change tends to which value?

Section B

Separable variable differential equations

In your A level course all of the differential equations are of one type. These are separable variable first order ordinary differential equations. 'First order' means you only have a term in the first derivative, not in the second or later derivatives. 'Ordinary' refers to the fact that you are using differentiation and not partial differentiation. 'Separable variable' means that you can manipulate the equation to store all of the expression in one variable on one side of the equation and anything in terms of the other variable on the other side.

Example:

A mug of tea cools at a rate proportional to its temperature. If it is initially at 80˚C and it takes 10 minutes to cool to half that temperature, how long does it take to cool to the room temperature of 20˚C? What is the limitation of this model? How could you make the model more realistic?

Resource Four
Data Source

Active Learning

Answer:

Here it is best to use θ to represent temperature, as T tends to get confused with the t for time. Remember the minus sign as the tea is cooling.

$$\frac{d\theta}{dt} \propto -\theta \Rightarrow \frac{d\theta}{dt} = -k\theta$$

Separating and integrating gives:

$$\int \frac{1}{\theta} d\theta = \int -kdt$$

So:

$$\ln\theta = -kt + c \Longrightarrow \theta = e^{-kt+c}$$

We have an initial condition, as we know θ when t=0. This means that c=ln80 and θ=80e^{-kt}

We also know that when t = 10, θ = 40°C. This gives us a value for k:

$$\ln 40 = -10k + \ln 80 \Longrightarrow k = \frac{1}{10}\ln 2 \Longrightarrow \theta = 80e^{-\frac{1}{10}(\ln 2)t}$$

So now we put this together to produce the time to cool to 20°C:

$$\ln 20 = -\frac{1}{10}\ln 2 \times t + \ln 80 \Longrightarrow t = (\ln 20 - \ln 80) \times -\frac{10}{\ln 2} = 10 \times \frac{\ln 4}{\ln 2} = 20\ minutes$$

As t tends to infinity, the model allows the temperature of the tea to tend to zero degrees which in reality would not happen.

If you set the rate of cooling proportional to the difference between the temperature of the tea and the temperature of the room, this would allow the tea to stop cooling once it reached room temperature.

Resource Four
Data Source

Activities

Exercise 2:

i. A population of fruit flies grows at a rate equal to 3 times the number of flies in the population at any one time, t, where time is measured in days. If there are originally P0 flies, how long does it take for the population to double?

ii. A second model for the same population of flies gives population growth being equal to 3cos3t times the number of flies in the population at time t. How long does it take the population to double now? Hint: Remember to use radians!

iii. Explain which model you think is better.

These questions reflect just one type of solvable differential equation. There are many different types of differential equation that can be solved analytically like these, but still more exist that require numerical solution. This is where calculus will not provide an answer and instead a method must be used to approach an answer step by step.

Section C

Numerical methods:

One of the simplest ways to do this is to apply the Euler's method. This uses something called a Taylor expansion:

$$f(x+\Delta x) = f(x) + \Delta x \frac{df(x)}{dx} + \frac{\Delta x^2}{2!}\frac{d^2f(x)}{dx^2} + \frac{\Delta x^3}{3!}\frac{d^3f(x)}{dx^3} + \cdots$$

Here Δx represents a small step taken in the x direction. You should be able to see what effect the size of Δx has on the number of terms of the expansion you'll need to take.

For the Euler method, we just use the first two terms:

$$f(x+\Delta x) = f(x) + \Delta x \frac{df(x)}{dx}$$

Resource Four
Data Source

So how does this help us to solve a differential equation? Well think of taking steps of length Δx through your x values to gradually approach an answer for $f(x)$. We do need one value for the function at one particular x just to start us off and we need to choose a step length. Then we will iterate using our newly calculated $f(x+\Delta x)$ value as the new f(x), until the answer appears to home in on a value.

Data Interpretation

Example:

Solve $\frac{df(x)}{dx} = \frac{\ln(f(x))}{4x}$, where $f(3) = 2$

Answer: Set up a table to record your results. We will take steps of 0.1 and see how many are needed.

x	$f(x)$	$\frac{df(x)}{dx}$
3	2	0.0577622…
3.1	2.0057762…	0.0561315…
3.2	2.0113893…	0.0545957…
3.3	2.0168489…	0.0531466…
3.4		

First put in your given values for x and $f(x)$. Use these to calculate $df(x)/dx$ using the given formula.

Now move x to 3.1 and use the equation above to find $f(3.1)$. So add 2 to 0.1 times $df(3)/dx$.

Put this new value back into the $df(x)/dx$ equation to get the new gradient.

Resource Four Data Source

Activities

Exercise 3:

Continue in this way until x=5 . Remember to save values on your calculator to ensure accuracy, or else use a spreadsheet. By plotting the points $(x, f(x))$ you will obtain a numerical solution to the differential equation, showing you the function $f(x)$ in the range 3 to 5.

Exercise 4:

i. Try this approach for $\frac{df(x)}{dx} = \frac{(f(x))}{x}$ with a starting value of $f(1) = 4$ for $1 \le x \le 2$. Use a step length of 0.1.

ii. What do you notice about the gradient?

iii. Solve the equation from i) analytically.

iv. What do you notice about your answers?

Exercise 5: (You will need a spreadsheet for this)

i. Try this approach for $\frac{df(x)}{dx} = \frac{f(x)}{e^x}$ with a starting value of $f(1) = 3$ for $1 \le x \le 3$. Use a step length of 0.1, then 0.01.

ii. Solve the equation in i) analytically and work out values for $1 \le x \le 3$, first in steps of 0.1, then in steps of 0.01.

iii. Plot the values from i) and ii) for the 0.1 step on one graph and for the 0.01 step on another.

iv. What do you notice about your 2 plots?

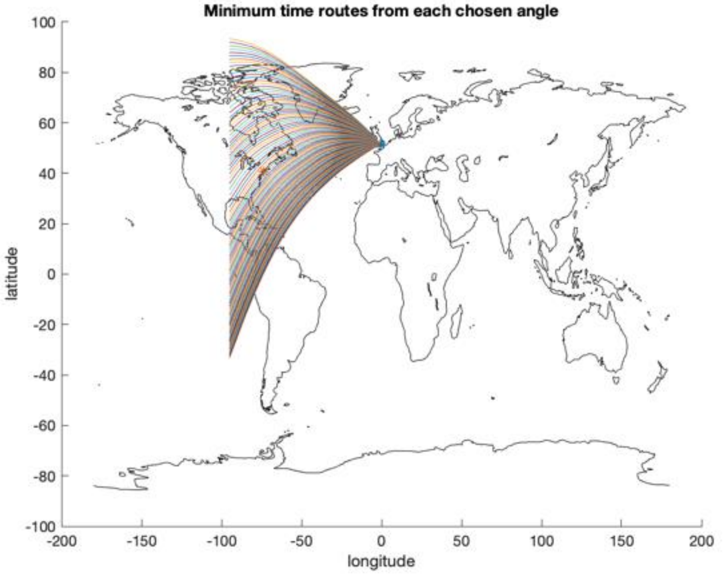
Resource Four
Data Source

Section D

Plane paths

Active Reading

Now we can apply this method to creating an aircraft trajectory. To start a minimum time trajectory an initial heading angle must be estimated. It is easiest to choose a range of angles and then use these to plot lots of trajectories until the flight path goes close enough to the destination airport.



We use the Euler method to convert our three differential equations that describe 2D flat plane flights to minimise time of travel:

$$\frac{dx}{dt} = W\cos\theta + u(x,y)$$
$$\frac{dy}{dt} = W\sin\theta + v(x,y)$$

$$\frac{d\theta}{dt} = sin^2\theta\frac{\partial v}{\partial x} - cos^2\theta\frac{\partial u}{\partial y} + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) sin\theta cos\theta$$

Resource Four
Data Source

into forward step equations, but we change to angles from distances as it will be easier to have points described in terms of longitude and latitude. As we flattened our grid and stretched the longitudes to give squares, we have to divide by the distance in metres that corresponds with a shift through 1 degree of latitude:

$$lon_{t+1} = lon_t + \frac{W \cos\theta_t \ + u}{111\ 000} \times dt$$

$$lat_{t+1} = lat_t + \frac{W \sin\theta_t + v}{111\ 000} \times dt$$

$$\theta_{t+1} = \theta_t + \frac{sin^2\theta_t \frac{\partial v}{\partial lon} - cos^2\theta_t \frac{\partial u}{\partial lat} + \left(\frac{\partial u}{\partial lon} - \frac{\partial v}{\partial lat}\right) sin\theta_t cos\theta_t}{111\ 000} \times dt$$

This means that if we have an initial angle and a time step, we can feed these, along with our initial position and the wind at that position, into the equations for the next longitude and latitude.

This takes us along our first step. Once we have a new position, we can use numerical gradients and interpolation (see Resource 3) to gain all of the information needed to generate the next angle. Now we repeat the process, recording all of the points passed through to get as close as possible to the destination airport.

Activities

Exercise 6:

i. What effect will changing the time step have?

ii. How can the time of flight be recovered from this method?

iii. Given that Heathrow has co-ordinates (51.5°N, 0.454 °W), that the initial heading angle is 187° , that the zonal wind speed is 20 m/s and the meridional wind speed is 5 m/s, using a 10 second time step and an air speed of 230 m/s, find the first position reached.

Resource Four
Data Source

iv. The gradient of the zonal wind at the starting airport is -0.4 in the direction of longitude and 0.3 in the direction of latitude. For the meridional wind these values are -2.7 and -3.2 respectively. What is the next heading angle?

v. How many steps of 10s is the journey likely to take if the plane is due to land in New York?

Once the minimum time paths for each air speed have been found for a particular day's wind field, the time can be multiplied by a value for the fuel burn rate for that model of aircraft at that speed.

This allows the best possible trajectory to be chosen to minimise fuel burn.

Resource Four
Further Reading

Explore

Independent Research

Now try looking at:

Differential equations:

- https://ibmathsresources.com/2014/02/28/differential-equations-in-real-life/
- http://tutorial.math.lamar.edu/Classes/DE/DE.aspx
- https://jmahaffy.sdsu.edu/courses/f00/math122/lectures/linear_diff_equation/linde_eq.html

Numerical methods:

- http://faculty.olin.edu/bstorey/Notes/DiffEq.pdf
- https://www.khanacademy.org/math/ap-calculus-bc/bc-differential-equations-new/bc-7-5/v/eulers-method

Aircraft and climate change:

- https://www.icao.int/Meetings/greenairports/Documents/3.%20Paul%20Williams.pdf
- https://www.iata.org/events/Documents/sfo2019-day3-climate-change.pdf

Resource Five Overview

Topic

Checking the Significance of Results

A-level Modules

Understand and apply the language of statistical hypothesis testing.

Understand that a sample is being used to make an inference about the population and appreciate that the significance level is the probability of incorrectly rejecting the null hypothesis.

Understand and use the Normal distribution as a model.

Recognise and interpret possible outliers in data sets and statistical diagrams.

Select or critique data presentation techniques in the context of a statistical problem.

Be able to clean data, including dealing with missing data, errors and outliers.

Objectives

After completing this resource, you should be able to:

- ✓ Analyse results critically.
- ✓ Classify and search for outliers.
- ✓ Test the significance of your results using a Student's T-test both by hand to gain a t value and on a spreadsheet to find the probability value associated with your data.

Instructions

1. read the source
2. answer the questions
3. explore the further reading
4. move on to the next resource

Resource Five
Data Source

Section A

Comparison with real world data

Once you have run your simulations and worked out how much fuel a plane would use following your new routes, that have been specially tailored to make the most of the available wind field, you need to ensure that you really are producing savings that will make the airlines take notice. The International Civil Aviation Organisation has issued a target that all airlines should become 2% more fuel efficient every year from now on. This is a big change and will particularly affect the legacy airlines, like British Airways, that are most likely to use whatever fuel is necessary to get passengers to their destinations on time. Low cost airlines, like Norwegian Air, tend to be a bit more fuel conscious, as this is one of their main expenses.

Section B

Dealing with data

It is really important that you find some real world data with which to compare your simulated flights. Luckily there are various websites that do provide at least some of the necessary information and still more can be modelled using what information is available. For example, to find an air speed, you can use the given ground speed of a plane and relevant wind field for the day that particular flight was in the air.

Once you have obtained all of your data you need to start comparing it.

Just looking at the figures is a good first approach.

Resource Five
Data Source

Activities

Data Interpretation

Exercise 1 :

On the next page are some results for looking at fuel used in simulations versus actual flights. The plane being used is a Boeing 787-9 and the real world data comes from a low cost airline.

i. How would you start to analyse the data?

ii. Are there any values of data that need "cleaning"? (Cleaning involves identifying and removing spurious values, but remember that outliers are often important to a data set, so should not be removed without comment. Ideally you need to analyse data both with and without outliers to see if they affect patterns adversely or not. Anomalies can occur due to error, but often are due to a change in a pattern. For example, in flight data, a journey could be longer if a plane has been diverted for some reason, but if the journey distance is very much longer, it may just be an error.)

iii. What further calculations would help you to argue your case if you wanted to prove airlines could save fuel by flying your new routes?

iv. How could you present your findings graphically to make your point?

Resource Five
Data Source

Fuel (kg)		Fuel (kg)	
Simulated flights	Real world data	Simulated flights	Real world data
35724.29748	37514.15575	35519.65863	38920.51302
38619.31587	40103.31898	35130.17112	38519.36692
37757.64777	38565.19376	34362.32868	38875.36032
37970.5371	38049.60359	34672.53	39695.6386
36738.28202	38049.60359	31789.4501	40971.22678
34313.22523	42242.0353	32157.19635	38179.0248
36140.70586	38132.3135	32637.6823	38476.5452
35080.28877	39630.94163	33152.15153	38479.08547
34879.91332	36892.37578	33103.42105	39364.48693
34748.1323	37049.43017	35297.65848	45443.19536
34529.43188	36505.10712	35583.16724	41093.31062
35480.14216	37549.89289	35471.64309	42964.54788
35292.10673	38021.70951	36182.52211	43748.95096
35505.54561	39666.91764	35266.76028	41320.07522
36450.47778	38278.44987	33148.96295	41408.43726
35414.61454	36249.41467	34768.28022	37864.31618
36254.82761	37776.70125	36110.23	39460.92335
35860.63307	37605.20125	35352.36231	38876.10781
35758.78842	41250.19213	35472.35	38577.68885
35448.40138	39753.21559	34658.40905	37694.49053
35158.89655	38143.50724	34438.19922	39811.5962
35979.27698	39064.08465	34997.39319	41256.76904
34642.98787	38015.01301	35938.81544	39817.78605
33689.32298	37037.25002	35933.17023	38683.55027
34087.82526	36992.09935	34939.61335	40242.17401
33466.90042	38003.73812	33938.16048	38549.52992
33013.66201	36730.46029	33922.9366	38105.87335
35192.12909	36954.93138	34917.30762	39818.45448
34389.02387	40199.41951	35830.01458	41597.34998
33731.2895	40449.1355	34626.16476	36685.58097
34581.51024	35669.8451	35248.88997	36376.83267
36157.67997	36253.95761	33922.39802	37279.04837
35549.79557	36716.36344	35216.0674	38990.67573
35344.20775	38426.21959	35230.14875	38194.70842
34994.99008	39374.95196	35417.43081	35709.89772
35139.29919	37844.72297	35306.18809	38051.64296

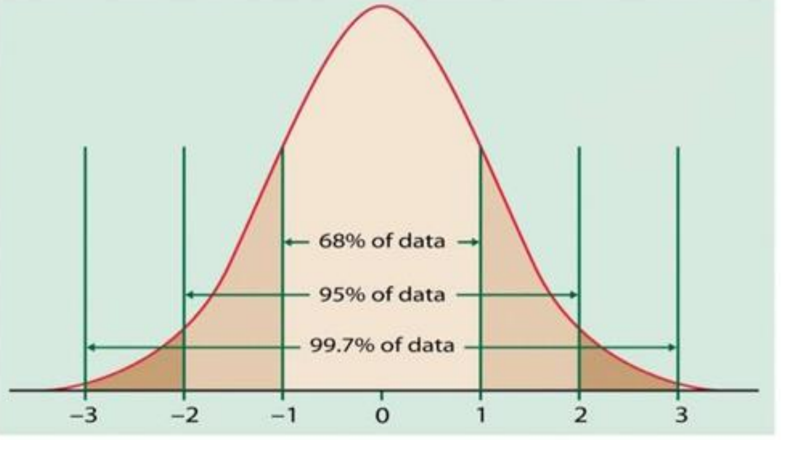
Resource Five
Data Source

Section C

Testing for significant differences between data sets

Active Reading

In trying to prove that your results are important, it is useful to look at their statistical significance. This is a measure of how confident you are that the change you are claiming is correct. With two sets of readings we can apply a Student's T-test. This tests whether there is a significant difference between the means of the two distributions. Although designed for normally distributed data (data that follows a bell shaped curve with two thirds of the data within one standard deviation of the mean), the T-test is robust to non-normal distributions too, meaning it works even if your data does not follow this shape.



To perform the test you need to have a null and an alternative hypothesis. The null hypothesis, says that nothing has changed, the alternative hypothesis includes the pattern you think might be applicable. Here the alternative would be that the simulated flights used less fuel on average than the real world flights. This is a one tailed test, as you are only checking one side of the distribution; you are checking for a decrease. In a two tailed test you would be checking for any difference, so would need to check both sides of the distribution. Your spreadsheet package will probably have a function called "t-test" which requires you to specify the two arrays of data to be tested, how many tails you are using and what type of test you need.

Resource Five
Data Source

As our data is about the same day's weather, and the same plane model just flying a different route between the same airports, we will need to perform a paired T-test. (If you are using Excel, this is defined to be a type 1 test.) However, before you try this, it is worth just carrying out a basic T-test on a smaller sample of data, so that you can understand fully what the spreadsheet has been programmed to do. The spreadsheet will return just a value for a probability which you will compare with your chosen significance level to see if there is a significant difference between the two sets of data or not. The test works out how likely it is that the difference you have seen happened by chance. So the smaller the resulting probability you get back, the more likely it is to be significant. Most of the time a 5% significance level is used, so you can be at least 95% confident that there is a change.

Example:

Here is a set of data giving the total number of incidents that airlines have recorded between 1985-1999 and then between 2000 and 2014. I have selected the sample randomly using systematic sampling from the full list of 57 companies. (You can find a link to the whole data set here:

https://github.com/fivethirtyeight/data/tree/master/airline-safety

We will need to use the paired T-test as the same airlines are being used for the data.

Resource Five
Data Source

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Airline	Number of incidents 1985-1999	Number of incidents 2000-2014
Air Canada	2	2
All Nippon Airways	3	7
China Airlines	12	2
Ethiopian Airlines	25	5
Japan Airlines	3	0
Malaysia Airlines	3	3
Saudi Arabian	7	11
TACA	3	3
US Airways/America West	16	11

Data Interpretation

We are going to look if there has been any improvement in safety, so we will test to see if the new data are significantly lower than the old data across this array.

To start we subtract the new data from the old data. Then we square these values and add up both column totals.

X	Y	X-Y =D	$(X-Y)^2=D^2$
2	2	0	0
3	7	-4	16
12	2	10	100
25	5	20	400
3	0	3	9
3	3	0	0
7	11	-4	16
3	3	0	0
16	11	5	25
Totals		$30=\sum D$	$566=\sum D^2$

Now we need to look at the T-score

which tells us the ratio of how different the differences between the groups are compared to how different the data within each group are. A large T-score means very different data between the groups, a small score means that the data are very similar.

Resource Five
Data Source

$$t = \frac{\frac{\sum D}{n}}{\sqrt{\frac{\sum D^2 - \frac{(\sum D)^2}{n}}{n(n-1)}}}$$

Here n is the number in our sample so we get:

$$t = \frac{\frac{\sum D}{n}}{\sqrt{\frac{\sum D^2 - \frac{(\sum D)^2}{n}}{n(n-1)}}} = \frac{\frac{30}{9}}{\sqrt{\frac{566 - \frac{30^2}{9}}{9 \times 8}}} = 1.310\ (to\ 3dp)$$

We now need to refer to the t distribution table that allows us to find the expected critical value for t at our chosen level of significance. Here we will use a 5% significance level (A). We also need to know the degrees of freedom of our data (DF). Degrees of freedom specify which of a range of curves best fits our data. The larger this value is, the closer the distribution gets to a normal distribution. In this case we have 9-1=8.

Looking along the DF=8 row and down the A=0.05 column gives a Critical t value of 1.860. The calculated value of 1.310 is lower than this, so the probability of the drop in incidents happening by chance is more than our chosen significance level of 5%. This means we must reject the alternative hypothesis and conclude that airlines have not improved their safety between the two fifteen year periods chosen.

DF	A = 0.1	0.05	0.025
∞	t_a = 1.282	1.645	1.960
1	3.078	6.314	12.706
2	1.886	2.920	4.303
3	1.638	2.353	3.182
4	1.533	2.132	2.776
5	1.476	2.015	2.571
6	1.440	1.943	2.447
7	1.415	1.895	2.365
8	1.397	1.860	2.306
9	1.383	1.833	2.262
10	1.372	1.812	2.228

Resource Five
Data Source

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Section D
Outlying values

But is all of the data valid? Should we first look at "cleaning" the data? As it is from a reliable source, we can be pretty sure that all of the values are correct. However, are some of them outliers that might be corrupting our answer? Well it is very important to acknowledge that outliers are still part of our data, so the best plan is to run our test both with and without any outlying values.

The two most useful tests for outliers are:

i. if you calculate the quartiles of your distribution, any value more than 1.5 times the interquartile range below the lower quartile or above the upper quartile is considered an outlier.

ii. you calculate the sample mean and standard deviation, then subtract the mean from each datum and divide by the standard deviation. If this value for a particular datum is more than 2, it can be classified as an outlier. However, you could choose a value other than 2, like 2.5 or even 3.5 depending on your particular set of data.

Activities

Problem Solving

Exercise 2:

i. Analyse the data for the air incidents to find any outliers using both of these methods.

ii. Try removing any outliers you find and recalculating the t value for the data.

iii. Is there any difference between your results?

iv. Try the spreadsheet ttest function on both the complete data set and the reduced data set to find the probabilities. Do the results agree with your previous conclusions?

Resource Five
Data Source

Activities

Exercise 3:

Now try to apply this theory to the set of data values given for fuel use for both the simulated and flown routes. This data compares the Boeing 787-9 Norwegian Air flights from London Gatwick Airport to John F. Kennedy Airport in New York to the simulations produced for flights on the same day in the same model of aircraft. What conclusions can you draw from your test?

You will need to refer to this t-test table as you are working with a larger data set.

http://www.meracalculator.com/math/t-distribution-critical-value-table.php

Resource Five
Further Reading

Explore

Independent Research

Now try looking at:
Hypothesis testing:

- https://machinelearningmastery.com/statistical-hypothesis-tests/
- https://www.statisticshowto.datasciencecentral.com/probability-and-statistics/t-test/

Identifying outliers:

- https://towardsdatascience.com/a-brief-overview-of-outlier-detection-techniques-1e0b2c19e561
- Dealing with outliers:
- https://www.geotab.com/blog/data-cleaning/
- https://www.rapidinsight.com/handle-outliers/
- https://www.theanalysisfactor.com/outliers-to-drop-or-not-to-drop/

Free data sets to practise on:

- https://www.dataquest.io/blog/free-datasets-for-projects/
- https://www.flightradar24.com

Resource Six Overview

Topic

Mathematical Research

A-Level Modules

Interpret and communicate solutions in the context of the original problem.

Understand the concept of a mathematical problem solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to repeat the cycle.

Construct and present mathematical arguments through appropriate use of diagrams; sketching graphs; logical deduction; precise statements involving correct use of symbols and connecting language, including: constant, coefficient, expression, equation, function, identity, index, term, variable.

Objectives

After completing this resource, you should be able to:

- ✓ Illustrate the importance of membrane-related research for the study of new medicines
- ✓ Evaluate the experimental results in a lipid-focused context.

Instructions

1. read the source
2. answer the questions
3. explore the further reading
4. move on to the next resource

Resource Six
Data Source

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Section A

Reading academic papers

Active Reading

One big change between studying mathematics at school and at university, is that at degree level, you will be expected to read research papers and letters and use the research of others to help your own work. Sometimes you might use a formula proved in a different paper, or else you might use some of the research presented to justify your own investigation. Papers you read will also lead you to other relevant papers and often you will break off looking at one to move to another that is mentioned in it. You will be encouraged to read at least a paper a week throughout your PhD studies and often by the time you finish your studies you will have written a couple academic papers yourself. Papers involving a lot of mathematics can be difficult to read, but here are some hints to help you if you would like to start researching in preparation for the next stage of your academic career. Start with an internet search on your favourite topic. This will probably take you to a Wikipedia page, but if you scroll to the base of this page you will find references and these are often excellent papers to start with. There are two types of paper, a Review Paper and a Research Paper. The Research Paper contains some new results, possibly based on previous research, whereas the Review Paper will be a round up of relevant research in one particular area. The title of the paper is your first clue as to whether this is the right read for you or not. However, mathematics re-uses many terms in a number of contexts, so you cannot always be sure that the paper you have selected will be relevant. Always note down the name and author of the paper, the publisher and the year of publication.

Resource Six
Data Source

The best place to start is the abstract. This is normally a 300 word summary of why the paper has been written and what conclusions can be drawn from the research. Quite often just reading this section is enough to tell you if you need to read the whole paper or not. Next comes the introduction, which will place the mathematics in the context of previous research. This is also a good source of more relevant papers. Note down the names of these and their authors so you can read them next, or sometimes before the paper you have in front of you. It is worth reading the introduction and then noting down whatever you remember from it. In this way you will avoid copying any key phrases if you decide to use your notes in a paper or project of your own. At this point it works to turn to the conclusion of the paper. Here you will find a longer summary of methods used, results gained and also normally some sort of idea for future research directions. These might justify your own work, so it is important to read any points given. If a particular quote would help in your own project, remember to note it down exactly, but put a link to the paper it comes from so that when you look back at your notes you remember those are someone else's ideas. You could even highlight quotes so they stand out more when you review your notes. Once you have finished with these sections, you will be better placed to see if it is worth reading the whole paper, or if you have enough detail already.

Activities

Choose a topic from the list below and try finding some relevant papers. Read through as discussed and take notes. You can then refer to these in university interviews and academics will be delighted that you have used your own initiative to explore Mathematics beyond the curriculum.

i. The Mathematics Behind Green Aviation

ii. Optimal Control Theory in Trajectory Planning

iii. Calculus of Variations in Optimising Fuel Burn for Aircraft

iv. Climate Change and Aviation

Resource Six Data Source

Academic Papers are very useful, as to be published these must have been peer reviewed, which means that other mathematiciansv in the same area of research as the author must have read the paper and agreed that it makes sense and is based on valid research. You will find articles about Mathematics in other sources too, like newspapers and online forums. It is best to be a little less trusting of these.

Section B

Presentations and posters

Active Reading

As a mathematical researcher you will need to attend conferences at which you will meet other mathematicians or scientists also interested in your area of work. At these conferences there are generally keynote speeches from leaders of the field, some shorter presentations that give everyone a chance to discuss their latest findings and poster sessions, where results are presented by way of academic posters. Creating a mathematical presentation is much like creating any sort of talk, but you just have to judge your audience a little more carefully. Are they likely to cope with the rigour of your proofs, or would they benefit more from just hearing about your results with a brief overview of your method? It really does depend on the conference delegates. There are no rules as to how many slides to include for how many minutes, but the more times you rehearse the slicker your performance will be. It is worth preparing slightly more than you think you need originally, but have slides that can be easily skipped over if you do run short of time. A mathematical presentation can be a bit dry if you are not careful, so try to think of good analogies you can use and remember to have some exciting images that can pop on the screen when least expected to keep your audience on its toes! It also helps to watch a few presentations mathematicians have given, for example some of the TED talks are a good reference point.

Resource Six
Data Source

Activities

Exercise 2:

i. Prepare a short presentation based on a paper you read in Exercise 1. Try recording it and watching it back. This is a really good way to make sure that you sound knowledgeable and convincing. Never put too much writing on slides and do always assume that your audience can read, so talk around your slides, highlighting your point, rather than reading them out.

ii. Look online for some relevant conferences. You should see details of the "call for abstracts". This is when academics are required to send in their 300 word summaries to pitch for a place presenting at the conference. Write an abstract for your presentation that would be suitable to send to the most relevant conference you find. You can be inspired by reading abstracts from other conferences. (This site might help: https://archive.siam.org/meetings/)

Preparing an academic poster is actually far harder than you might expect. Having had it drummed into you that you must reference every last part of your dissertation and show full working at all times, creating a poster can come as a bit of a shock! What is important here are your results, so the framework of your poster should include aims and motivation, a very brief description of your method and then lots of lovely big diagrams showing your results and conclusions. You have to remember that your poster will be in a room with possibly hundreds of others, so if you want to attract a few people over to talk about the nitty gritty mathematics behind the work, you need to think about large colourful images.

Resource Six
Data Source

Activities

Exercise 3:

Here is my first poster, about the spherical version of the work completed giving new trajectories for planes. I was quite pleased with it, until I reached the conference and realised that there were some reasons why it wasn't as exciting as some of the posters around me. Read through it carefully and make a list of its good points and any improvements that would have made it stand out more.

Resource Six Data Source

Motivation: A better future and an economically viable industry

- About $\frac{1}{20}$ of all anthropogenic climate change comes from aviation[1].
- 859 million tonnes of CO_2 were emitted by this industry in 2017[2].
- 24.2% of airline operating costs estimated to be spent on fuel in 2019[3].

Aim:

Show that routing to take advantage of the wind field saves fuel and thus benefits both the economy and the environment.

Modelling assumptions:

1. Route between John F. Kennedy Airport, New York (JFK) and London Heathrow Airport (LHR).
2. Data about actual flights BA177 (LHR to JFK) and BA174 (JFK to LHR) from https://www.flightradar24.com.
3. Winter period from 1st December, 2018 to 28th February, 2019 considered.
4. Constant altitude of 35 000 ft and constant air speed.
5. Wind fields at pressure level of 250 hPa used from National Center for Atmospheric Research re-analysis dataset[4].
6. Fuel burn calculations for Boeing 747-436 and 777-236 completed using new method[5].
7. Heading angles varied to give shortest time path for each wind field in each direction.

Dynamical system:

The spherical simulation data was obtained by adapting Zermelo's equations[6] as shown in Arrow's 1949 paper [7]. The equations thus derived are:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{u + V\cos\theta}{R\cos y} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{v + V\sin\theta}{R} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{Wind3D}{R\cos y}$$

x and y are distances calculated from the position, V is air speed, θ is heading angle and R is the radius of the Earth, 6371 km. The $Wind3D$ term is derived as:

$$Wind3D = -\sin\theta\cos\theta\frac{\partial u}{\partial x} + u\cos^2\theta\sin y + \cos^2\theta\cos y\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + v\sin\theta\cos\theta\sin y + \cos\theta\sin\theta\cos y\frac{\partial v}{\partial y} + V\cos\theta\sin y + \cos^2\theta\frac{\partial v}{\partial x}$$

Numerical solution:

- Euler forward step method with a 1 s time step applied to system of equations.
- Initial heading angles taken as 25° either side of the great circle route (GCR).
- Method of bisection applied to initial heading angle until route passes within 200 m of destination airport.
- Fuel use for every speed from 200 to 260 m s^{-1} at 5 m s^{-1} intervals used to narrow search.
- Finally optimal air speed for minimum fuel use obtained to nearest 0.5 m s^{-1}.

Simulated paths:

- Paths simulated for all days across a range of speeds.
- Optimal air speed to reduce fuel used found by combining flight times and fuel burn rate calculations.

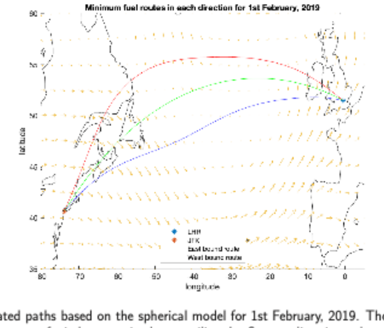
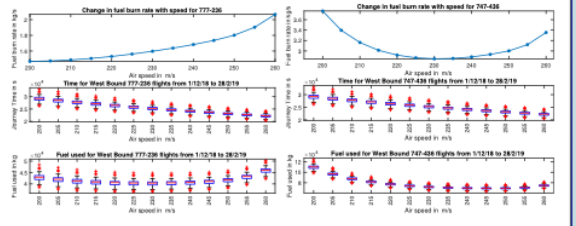


Figure 1: Simulated paths based on the spherical model for 1st February, 2019. The East bound route takes advantage of wind vectors in the prevailing Jet Stream direction, whereas the West bound route avoids these areas by flying further North. The green curve is the GCR.

Results for minimum fuel use:

Figure 2: Graphs to show how fuel burn, time of flight and fuel used change with air speed across the 90 days of simulations, flying from LHR to JFK for the Boeing 777-236 and 747-436 models.



- Air speed for lowest fuel burn rate lower than air speed for lowest fuel use.
- East bound results similar, but with around 10% less fuel used.

Comparison with flight data:

- Data from the simulation and the logged BA174 and BA177 flights were compared.

Direction	Plane	Total fuel saving (kg)	Average fuel saving (kg)	Average % fuel saving	No. of flights
E	747-436	120 000	6 200	8.0	20
E	777-236	25 000	2 100	5.4	12
W	747-436	330 000	15 000	17	20
W	777-236	41 000	3 700	8.4	11

Table 1: Table showing fuel savings made by flying optimal routes, between JFK and LHR in each direction when simulated routes were compared with recorded flight data.

- Full data for all flights was not available, so a model was created to find missing air speeds.

Direction	Plane	Total fuel saving (kg)	Average fuel saving (kg)	Average % fuel saving	No. of flights
E	747-436	330 000	9 100	12	36
E	777-236	91 000	2 600	6.6	35
W	747-436	530 000	14 000	16	39
W	777-236	61 000	3 600	8.0	35

Table 2: Table showing fuel savings made by flying optimal routes, between JFK and LHR in each direction when simulated routes were compared with a combination of recorded and modelled data.

Conclusions:

1. Allowing planes to fly according to the wind field was shown to give a **5.4 to 17% saving** compared with actual flight data.
2. A Student's T-test showed that there was a **significant decrease in fuel** use across the simulated routes compared with the combined flight and model data.
3. For the 747-436 the route flown was the biggest contributor to the saving made, but for the 777-236 the change to the air speed made the most difference.
4. On average **savings of 11% of fuel** could be made.
5. Based on these results approximately **58 million kg of carbon dioxide emissions could be avoided** and **$12 million saved** just on this one route across the winter season.

References:

[1] Lee, D et al, Atmospheric Environment, Vol 43, Pages 3520-3537, 2009.

[2] IATA *Climate Change Fact Sheet*, 2018a.

[3] IATA *Cautious Optimism Extends into 2019* Press Release No. 72, 2018b.

[4] Kalnay, E et al, Bull. Amer. Meteor. Soc., Vol 77, Pages 437-471,1996.

[5] Poll, D I A *A new method for the estimation of cruise fuel burn rate, thrust, engine efficiency and optimum altitude for high speed civil transport aircraft*, 2019, (Personal communication).

[6] Zermelo, E, Zeitschrift für Angewandte Mathematik und Mechanik, 11 (2), 1931.

[7] Arrow, K, Journal of Meteorology, Vol 6, No 2, Pages 150-159, 1949.

Resource Six
Further Reading

Explore

Now try looking at:

Reading papers:

- http://eprints.maths.manchester.ac.uk/2484/1/reading.pdf

Writing papers:

- https://people.bath.ac.uk/mamamf/talks/awayday_may2010.pdf

Creating academic posters:

- https://mathoverflow.net/questions/159305/what-is-a-good-poster-for-a-math-conference
- https://guides.nyu.edu/posters

Academic presentations:

- https://www.uts.edu.au/current-students/support/helps/self-help-resources/presentation-skills/conference-presentations

Independent Research

Final Reflection

Deep Dive

Now that you have all of the skills necessary to run a flight trajectory simulation, try accessing the wind data for a particular day from

https://www.esrl.noaa.gov/psd/data/composites/day/

To keep things reasonably simple, you will be using the compressed 2D flat grid from Resource 1, so your time results will be rather longer than actual flight times.

You will need to choose a pressure level. Either 250mb or 200mb would be reasonable for the cruise phase of a transatlantic flight, as the first corresponds roughly to an altitude of 35 000 feet and the second to an altitude of about 39 000 feet. Download the data for both the zonal and meridional winds.

Choose two airports either side of the North Atlantic and find out the co-ordinates of their positions.

Interpolate to find the wind speed in each direction at your departure airport using the linear technique from Resource 3.

Choose an initial heading angle, which appears to go roughly towards your chosen destination airport.

As you are using a flat grid, you can use trigonometry to find a good first approximation.

Choose an air speed that is between 230 and 250 m/s and a time step which will need to be no larger than 100s. (If you are programming code to automate this process, go straight for a 1s time step, as this will help you to approach the destination airport more closely.

A good start for mathematical programming is Octave which can be downloaded for free here: https://www.gnu.org/software/octave/

Now substitute each of these values into the numerical approximations, found in Resource 4, for the dynamical system derived in Resource 2:

$$lon_{t+1} = lon_t + \frac{W\cos\theta_t \ + u}{111\ 000} \times dt$$

$$lat_{t+1} = lat_t + \frac{W\sin\theta_t + v}{111\ 000} \times dt$$

Final Reflection

Deep Dive

Using the linear interpolation method and numerical gradients as discussed in Resource 3, find all of the information necessary to compute the next heading angle using:

$$\theta_{t+1} = \theta_t + \frac{sin^2\theta_t \frac{\partial v}{\partial lon} - cos^2\theta_t \frac{\partial u}{\partial lat} + \left(\frac{\partial u}{\partial lon} - \frac{\partial v}{\partial lat}\right) sin\theta_t cos\theta_t}{111\ 000} \times dt$$

Now use the first two differential equations to generate the position of the next point on your journey. Once there, interpolate and use numerical gradients again to give you your next heading angle. Continue in this way across the Atlantic. You will not necessarily reach your destination airport, but should check, using Pythagoras' Theorem how close you are at each stage. Once you start moving further from this point, it is not worth taking any more steps. To get closer, experiment with different initial heading angles.

Once you have a route that gets close to the destination, try plotting it on a map, but you will have to stretch to a flat grid.

This method is worked through for a single step at the end of Resource 4.

Now think about the clearest way to present your findings.

Choose one of the key ways to present research in mathematics and use it to explain your ideas to others. You'll find all of the material about these in Resource 6

Part 3 – Study Skills, Tips & Guidance

This section includes helpful tips to help you complete this pack, as well as improve your study skills for any courses you take next year.

It also includes a few fantastic easy-to-use resources to know what to do next if you are hoping to go to university in the next few years, like UCAS advice and web links to more academic opportunities.

In this section:

University Study Skills:
- ✓ Cornell Notes
- ✓ Key Instruction Words
- ✓ Academic Writing
- ✓ Referencing
- ✓ Evaluating Your Sources

University Guidance:
- ✓ What next?

Subject Guidance:
- ✓ More on studying your subject

University Study Skills
Cornell Notes

University Ready!

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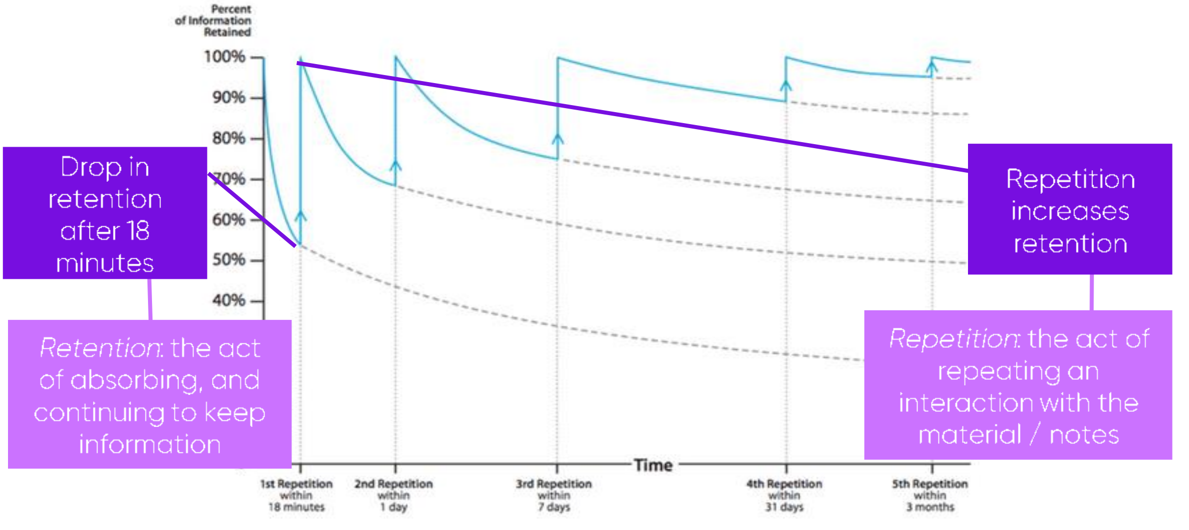
Why is good note taking important?

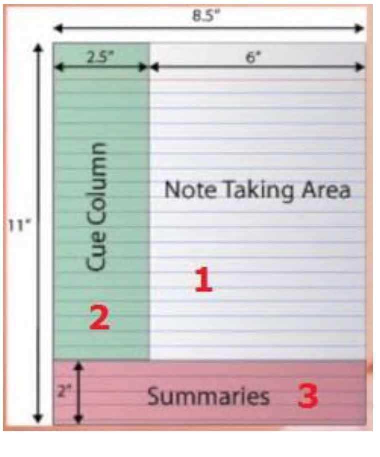
If it feels like you forget new information almost as quickly as you hear it, even if you write it down, that's because we tend to lose almost 40% of new information within the first 24 hours of first reading or hearing it.

If we take notes effectively, however, we can retain and retrieve almost 100% of the information we receive. Consider this graph on the rate of forgetting with study/repetition:



Learning a new system

The Cornell Note System was developed in the 1950s at the University of Cornell in the USA. The system includes interacting with your notes and is suitable for all subjects. There are three steps to the Cornell Note System.



Step 1: Note-Taking

1. Create Format: Notes are set up in the Cornell Way. This means creating 3 boxes like the ones on the left. You should put your name, date, and topic at the top of the page.

2. Write and Organise: You then take your notes in the 'note taking' area on the right side of the page. You should organise these notes by keeping a line or a space between 'chunks' /main ideas of information. You can also use bullet points for lists of information to help organise your notes.

University Study Skills
Cornell Notes

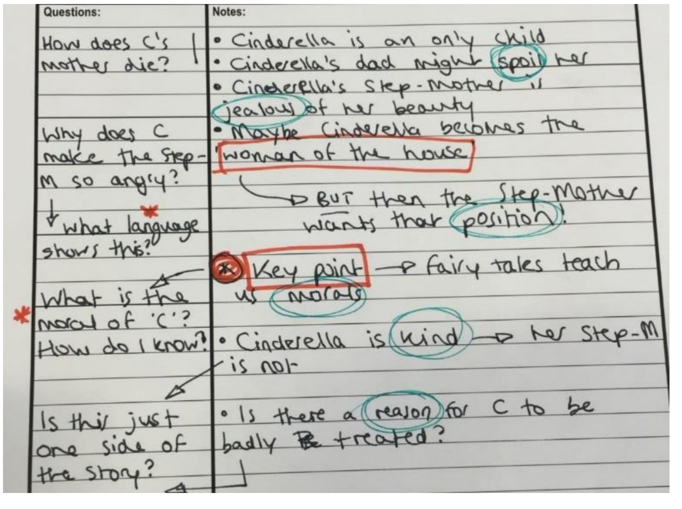
Step 2 Note-Making

1. Revise and Edit Notes: Go back to box 1, the note taking area and spend some time revising and editing. You can do this by: highlighting 'chunks' of information with a number or a colour; circling all key words in a different colour; highlighting main ideas; adding new information in another colour

2. Note Key Idea: Go to box 2 on the left hand side of the page and develop some questions about the main ideas in your notes. The questions should be 'high level'. This means they should encourage you to think deeper about the ideas. Example 'high level' questions would be:

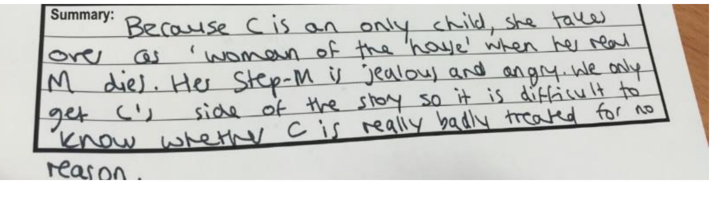
- Which is most important / significant reason for…
- To what extent…
- How does the (data / text / ideas) support the viewpoint?
- How do we know that…

Here is an example of step 1 and step 2 for notes on the story of Cinderella:



Step 3 Note-Interacting

1. Summary: Go to box 3 at the bottom of the page and summarise the main ideas in box 1 and answer the essential questions in box 2.



Give the Cornell Note Taking System a try and see if it works for you!

University Study Skills
Key Instruction Words

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These words will often be used when university tutors set you essay questions - it is a good idea to carefully read instruction words before attempting to answer the question.

Analyse – When you analyse something you consider it carefully and in detail in order to understand and explain it. To analyse, identify the main parts or ideas of a subject and examine or interpret the connections between them.

Comment on – When you comment on a subject or the ideas in a subject, you say something that gives your opinion about it or an explanation for it.

Compare – To compare things means to point out the differences or similarities between them. A comparison essay would involve examining qualities/characteristics of a subject and emphasising the similarities and differences.

Contrast – When you contrast two subjects you show how they differ when compared with each other. A contrast essay should emphasise striking differences between two elements.

Compare and contrast – To write a compare and contrast essay you would examine the similarities and differences of two subjects.

Criticise – When you criticise you make judgments about a subject after thinking about it carefully and deeply. Express your judgement with respect to the correctness or merit of the factors under consideration. Give the results of your own analysis and discuss the limitations and contributions of the factors in question. Support your judgement with evidence.

Define – When you define something you show, describe, or state clearly what it is and what it is like, you can also say what its limits are. Do not include details but do include what distinguishes it from the other related things, sometimes by giving examples.

Describe – To describe in an essay requires you to give a detailed account of characteristics, properties or qualities of a subject.

Discuss – To discuss in an essay consider your subject from different points of view. Examine, analyse and present considerations for and against the problem or statement.

University Study Skills
Key Instruction Words

Con't

Evaluate – When you evaluate in an essay, decide on your subject's significance, value, or quality after carefully studying its good and bad features. Use authoritative (e.g. from established authors or theorists in the field) and, to some extent, personal appraisal of both contributions and limitations of the subject. Similar to **assess.**

Illustrate – If asked to illustrate in an essay, explain the points that you are making clearly by using examples, diagrams, statistics etc.

Interpret – In an essay that requires you to interpret, you should translate, solve, give examples, or comment upon the subject and evaluate it in terms of your judgement or reaction. Basically, give an explanation of what your subject means. Similar to **explain.**

Justify – When asked to justify a statement in an essay you should provide the reasons and grounds for the conclusions you draw from the statement. Present your evidence in a form that will convince your reader.

Outline – Outlining requires that you explain ideas, plans, or theories in a general way, without giving all the details. Organise and systematically describe the main points or general principles. Use essential supplementary material, but omit minor details.

Prove – When proving a statement, experiment or theory in an essay, you must confirm or verify it. You are expected to evaluate the material and present experimental evidence and/or logical argument.

Relate – To relate two things, you should state or claim the connection or link between them. Show the relationship by emphasising these connections and associations.

Review – When you review, critically examine, analyse and comment on the major points of a subject in an organised manner

University Study Skills
Academic Writing

What is academic writing?

'Academic writing' is a specific way of writing when communicating research or discussing an argument/point of view. It has a logical structure, and it uses formal language. There is a particular tone, 'voice' and style to the language. Unlike creative or narrative writing, academic writing will also use different sources of information to support what is being said.

The language of academic writing: do's and don'ts

- Do use words you know the meaning of and are confident using, it doesn't have to be complicated to be clear!
- Do not use contractions; don't, can't, doesn't, it'd. Do write out fully; do not, cannot, does not, it would.
- Do not use colloquialisms- this is 'writing as you speak'. Examples include misuse of the words 'literally' or 'basically', common phrases, such 'like chalk and cheese'.
- Do not use slang or jargon. For example, 'awks', 'lit', 'woke'.

Expressing your opinion in academic writing

In academic writing, it is best practice to express an opinion without writing in the first person, which can often be challenging. Always bear in mind that your work should read like a voice that is guided by the evidence and not basic personal intuition.

Therefore, rather than saying 'In my opinion, this proves that', you can express the outcome of your reasoning in other ways:

- 'This indicates that...';
- 'The aforementioned problems in Smith's argument reveal that...';
- 'Such weaknesses ultimately mean that...', and so on.

Signposting

Signposting guides your reader through different sections of your writing. It lets those who read your writing know what is being discussed and why, and when your piece is shifting from one part to another. This is crucial to for clear communication with your audience.

Signposting stems for a paragraph which expands upon a previous idea	Signposting stems for a paragraph which offers a contrasting view
Building on from the idea that ... (mention previous idea), this section illustrates that ... (introduce your new idea).	However, another angle on this debate suggests that ... (introduce your contrasting idea)
To further understand the role of ...(your topic or your previous idea) this section explores the idea that ... (introduce your new idea)	In contrast to evidence which presents the view that ... (mention your previous idea) an alternative perspective illustrates that ...
Another line of thought on ... (your topic or your previous idea) demonstrates that ...	However, not all research shows that ... (mention your previous idea). Some evidence agrees that ...

University Study Skills
Referencing

University Ready!

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What is a reference or referencing?

A reference is just a note in your assignment that tells your reader where particular ideas, information or opinions that you have used from another source has come from. It can be done through 'citations' or a 'bibliography'.

When you get to university, you will need to include references in the assignments that you write. As well as being academic good practice, referencing is very important, because it will help you to avoid plagiarism.

Plagiarism is when you take someone else's work or ideas and pass them off as your own. **Whether plagiarism is deliberate or accidental, the consequences can be severe.** You must be careful to reference your sources correctly.

Why should I reference?

Referencing is important in your work for the following reasons:

- It gives credit to the authors of any sources you have referred to or been influenced by.
- It supports the arguments you make in your assignments.
- It demonstrates the variety of sources you have used.
- It helps to prevent you losing marks, or failing, due to plagiarism.

When should I use a reference?

You should use a reference when you:

- Quote directly from another source.
- Summarise or rephrase another piece of work.
- Include a specific statistic or fact from a source.

University Study Skills
Referencing

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Is it a source worth citing?

Question your sources before referencing using these tips:

Currency: the timelines of the information

- When was it published or posted? Has it been revised or updated? Does your topic require current information, or will older sources work as well?

Relevancy: the importance of the information for your needs

- Does the information relate to your topic or answer your question? Who is the intended audience? Have you looked at a variety of sources?

Authority: the source of the information

- Who is the author/publisher/source/sponsor? What are the author's credentials? Is the author qualified to write on the topic?

Accuracy: the reliability and correctness of the source

- Is the information supported by evidence? Has the information been reviewed or refereed? Can you verify whether it is a personal or professional source? Are there errors?

Purpose: the reason the information exists

- Does the author make the intensions/ purpose clear? Is the information fact opinion or propaganda? Are there are biases? Does the viewpoint appear objective?

University Study Skills
Referencing

How do I reference?

- There are a number of different ways of referencing, but most universities use what is called the Harvard Referencing Style. Speak with your tutor about which style they want you to use, because the most important thing is you remain consistent!

- The two main aspects of referencing you need to be aware of are:

1. In-text citations

- These are used when directly quoting a source. They are located in the body of the work, after you have referred to your source in your writing. They contain the surname of the author of the source and the year it was published in brackets.
    - E.g. *Daisy describes her hopes for her infant daughter, stating "I hope she'll be a fool—that's the best thing a girl can be in this world, a beautiful little fool."* *(Fitzgerald, 2004).*

2. Bibliography

- This is a list of all the sources you have referenced in your assignment. In the bibliography, you list your references by the numbers you have used and include as much information as you have about the reference. The list below gives what should be included for different sources.

- Websites – Author (if possible), *title of the web page*, 'Available at:' website address, [Accessed: date you accessed it].
    - E.g. *'How did so many soldiers survive the trenches?'*, Available at: http://www.bbc.co.uk/guides/z3kgjxs#zg2dtfr [Accessed: 11 July 2019].

- Books – Author surname, author first initial, (year published), *title of book*, publisher
    - E.g. Dubner S. and Levitt, S., (2007) *Freakonomics: A Rogue Economist Explores the Hidden Side of Everything*, Penguin Books

- Articles – Author, *'title of the article'*, where the article comes from (newspaper, journal etc.), date of the article.
    - E.g. Maev Kennedy, *'The lights to go out across the UK to mark First World War's centenary'*, The Guardian Newspaper, 10 July 2014.

University Study Skills
Evaluating your sources

Knowing about the different types of sources and what makes them worth using is important for academic work.

When doing research you will come across a lot of information from different types of sources. How do you decide which source to use? From newspaper articles to books to tweets, this provides a brief description of each type of source, and breaks down the factors to consider when selecting a source.

Tweets

A platform for millions of very short messages on a variety of topics.

Blogs

Blogs (e.g. Tumbler) are an avenue for sharing both developed and unpublished ideas and interests with a niche community.

Youtube

A collection of millions of educational, inspirational, eye-opening and entertaining videos.

Newspaper

A reporting and recording of cultural and political happenings that keeps the general public informed. Opinions and public commentaries can also be included.

Journals

A collection of analytics reports that outline the objectives, background, methods, results and limitations of new research written for and by scholars in a niche field.

Academic book

The information presented is supported by clearly identified sources. Sometimes each chapter has a different author.

Encyclopaedia

Books or online – giving information on many different subjects. Some are intended as an entry point into research, some provide detailed information and onwards references.

Popular magazine

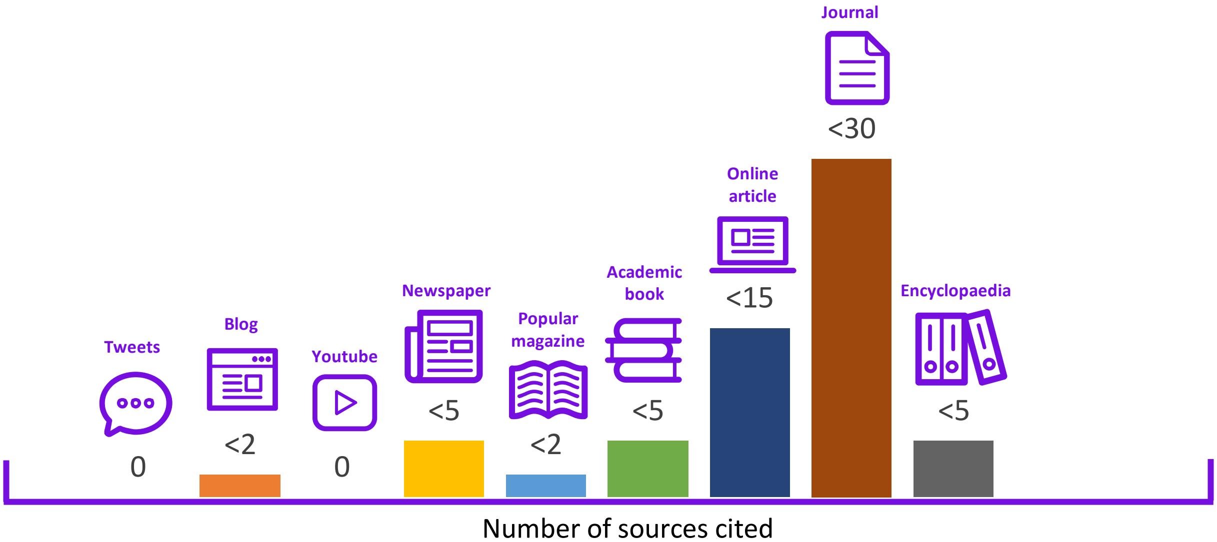
A glossy compilation of stories with unique themes intended for specific interests.

University Study Skills
Evaluating your sources

University Ready!

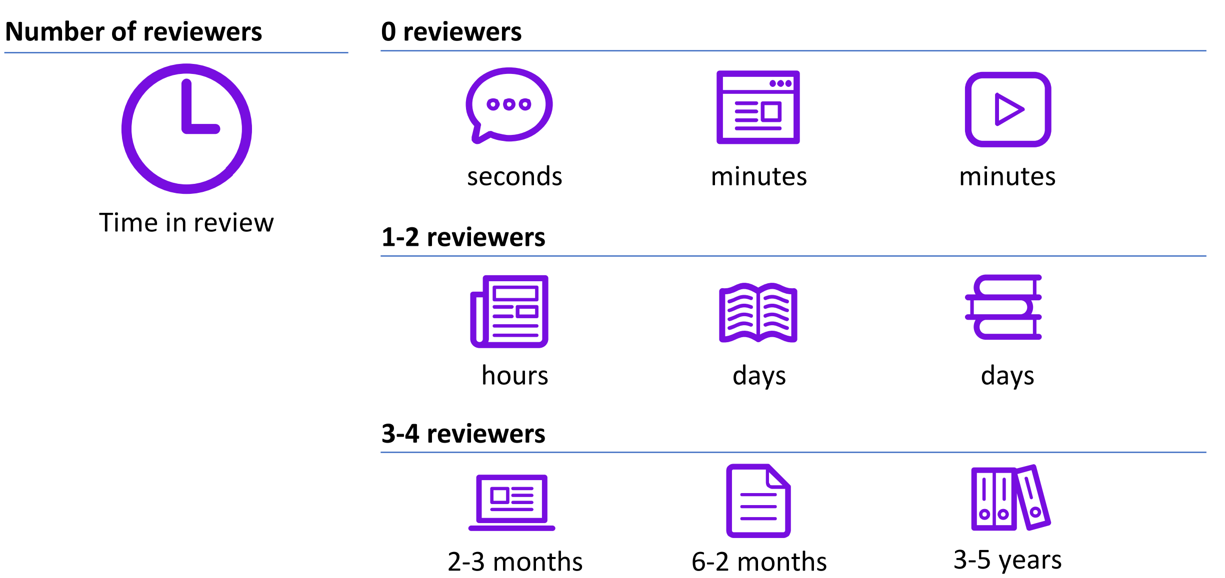
Number of outside sources

When an author used many outside sources into their writing, they demonstrate familiarity with ideas beyond their own. As more unique viewpoints are pulled into a source, it becomes more comprehensive and reliable. This shows the typical number of outside sources used in each publication.



Degree of review before a source is published

Two factors contribute to the amount of inspection that a source receives before it might be published: the number of reviewers fact-checking the written ideas, and the total time spent by reviewers as they fact-check. The more people involved in the review process and the longer the review process takes, the more credible the source is likely to be.



University Guidance
What next?

University Ready!

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University Guidance

Different people go to university for different reasons. You might have a particular job in mind or just want to study a subject you are passionate about. Whatever your motivations, going to university can help improve your career prospects, as well as develop your confidence, independence and academic skills.

Choosing a course and university

Choosing the right course to study is an important decision so make sure you take time to research the different options available to you. Here are some top tips:

- ✓ You don't have to choose a course which you have already studied, there are lots of courses which don't require prior knowledge of the subject. You can apply skills gained from school studies to a new field.
- ✓ The same subject can be taught very differently depending on the course and university you choose. Take a look at university websites to find out more about the course content, teaching styles and assessment types.
- ✓ When choosing a university, think about what other factors are important to you. Do you want to study at a campus university or be based in a city centre? What accommodation options are there? Does the university have facilities for any extracurricular activities you're involved in?
- ✓ To research your options, have a look at university prospectuses and websites, as well as seeing if there are opportunities to speak to current students who can give you a real insight in to what life is like there.

Insight into: University of Reading

University of Reading

The author of this coursebook attends the University of Reading.

The University of Reading runs a large number of sessions to help find out more about the process of applying to university as well as taster sessions and Open Online Courses in a number of different subjects. To find out more, visit: www.reading.ac.uk/virtual-events.

Chat to current University of Reading students via Unibuddy and get their views on what university life is like!

University Guidance
What next?

University Ready!

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Exploring Careers and Subject Options

- ✓ Find job descriptions, salaries and hours, routes into different careers, and more at https://www.startprofile.com/
- ✓ Research career and study choices, and see videos of those who have pursued various routes at http://www.careerpilot.org.uk/
- ✓ See videos about what it's like to work in different jobs and for different organisations at https://www.careersbox.co.uk/
- ✓ Find out what different degrees could lead to, how to choose the right course for you, and how to apply for courses and student finance at https://www.prospects.ac.uk/
- ✓ Explore job descriptions and career options, and contact careers advisers at https://nationalcareersservice.direct.gov.uk/
- ✓ Discover which subjects and qualifications (not just A levels) lead to different degrees, and what careers these degrees can lead to, at http://www.russellgroup.ac.uk/media/5457/informed-choices-2016.pdf

Comparing Universities

Use our platform ThinkUni.org to take a short quiz about your preferences and interests to find out which universities might be a great fit for you.

Other popular resources:

- ✓ https://www.ucas.com/
- ✓ https://www.whatuni.com/
- ✓ http://unistats.direct.gov.uk/
- ✓ https://www.thecompleteuniversityguide.co.uk/
- ✓ https://www.opendays.com/

University Guidance
What next?

University Ready!

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UCAS and the university application process

All applications for UK degree programmes are made through UCAS. There is lots of information on the UCAS website to guide you through the process and what you need to do at each stage.

Apply

- Applications **open in September** the year before you plan to start university.
- You can apply for up to **five courses.**
- The deadline for most courses is **15 January**, though there is an earlier deadline of **15 October** for Oxford and Cambridge, medicine, veterinary medicine/science and dentistry.

Decisions

- Some courses may require an interview, portfolio or admissions test in addition to UCAS application. Check individual university websites details.
- Check UCAS Track which will be updated with decisions from the universities you have applied for and to see your deadline for replying to any offers.
- You should choose a firm (or first) choice university and an insurance choice. If you already have your exam results or a university thinks your application is particularly strong, you might receive an **unconditional offer**.

Results

- If you're holding a conditional offer then you will need to wait until you receive your exam results to have your place confirmed.
- Clearing & Adjustment allows you to apply to courses which still have vacancies if you didn't meet the conditions of your offer, have changed your mind about what or where you want to study, or have met and exceeded the conditions of your offer and would like to look at alternate options.

Personal statements

A really important part of your application is the personal statement. The personal statement gives you the opportunity to tell universities why they should offer you a place.

Here a few top tips for making your personal statement stand out:

- You can only submit one personal statement so it's important that you are consistent in your course choices. Make sure you have done your research to show your understanding of the subject area and passion for it.
- Start by brainstorming all your skills, experience and attributes. Once you have everything written down, you can begin to be selective – you only have 47 lines so won't be able to include everything.
- The ABC method: action, benefit and course can be a useful way to help demonstrate your relevant experience and how it applies to the course you're applying for.

University Guidance
What next?

Personal Statement do's and don'ts

Read the tips below from real life professors and admissions staff in university Biology and Psychology departments, on the 'do's' and 'don'ts' of what to include in your personal statement:

Maths

- Passion for Mathematics – why you love it, what aspects do you enjoy and what motivates you to pursue Maths at University. Can you demonstrate this through your academic studies or extra-curricular activities?
- Which areas of Mathematics are you most interested in?
- Demonstrate skills such as problem solving , clear writing and attention to detail
- What aspects of studying mathematics at university are you looking forward to?
- Have you read any books about mathematics beyond those you use at school?

Further useful resources

Be sure you know what you'll need to do to apply to university in the UK:

- ✓ Key dates and deadlines: www.access-ed.ngo/timelines-for-applying-to-university
- ✓ Get tutor advice on writing a UCAS personal statement at www.access-ed.ngo/writing-your-ucas-personal-statement
- ✓ An easy template to start practising your personal statement: https://www.ucas.com/sites/default/files/ucas-personal-statement-worksheet.pdf
- ✓ Untangle UCAS terminology at https://www.ucas.com/corporate/about-us/who-we-are/ucas-terms-explained
- ✓ Discover more about the application process including when to apply and how to fill in your application on the UCAS website.
- ✓ Read more useful advice about what to include in your personal statement on UCAS, the Complete University Guide and The Student Room.
- ✓ Attend one of our virtual sessions to find out more about applying and personal statements.

Subject Guidance

University Ready!

UCAS

A Deeper look....

- ✓ **Read:** Try reading the journal of the Institute of Mathematics and its Applications, the SIAM journal or any other Mathematics journals that are available in your school library. These have shorter easier to access ideas than tackling whole papers and there is a large variety of material in each issue. There are also some great books out there, like Matt Parker's "Things to Make and Do in the Fourth Dimension" and "Humble Pi" or Marcus du Sautoy's "The Music of the Primes". For a good idea of what life as a professional Mathematician is like try "Birth of a Theorem" by Cedric Villani. For recreational Mathematics try anything by Ian Stewart.

- ✓ **Watch:** Watch some of the fabulous TED talks on Mathematics available at https://www.ted.com/topics/math. There are a huge variety of topics on offer, so just try a few and if you find one you particularly enjoy, try an online search for some papers about that specific field of the subject.

- ✓ **Listen:** Listen to podcasts such as My Favourite Theorem, The Secrets of Mathematics or Relatively Prime. These are all interesting in their own right and dipping in and out of them you are bound to be inspired.

- ✓ **Do:** Try to get along to some live Mathematics discussions or seminars. You can start with some lighter material by going to see Matt Parker or Marcus du Sautoy, but also try some of the open seminars available at your local university. These can all be found online and they give you a great opportunity to talk Mathematics with a whole range of people.

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