IB Mathematics SL Study Guide

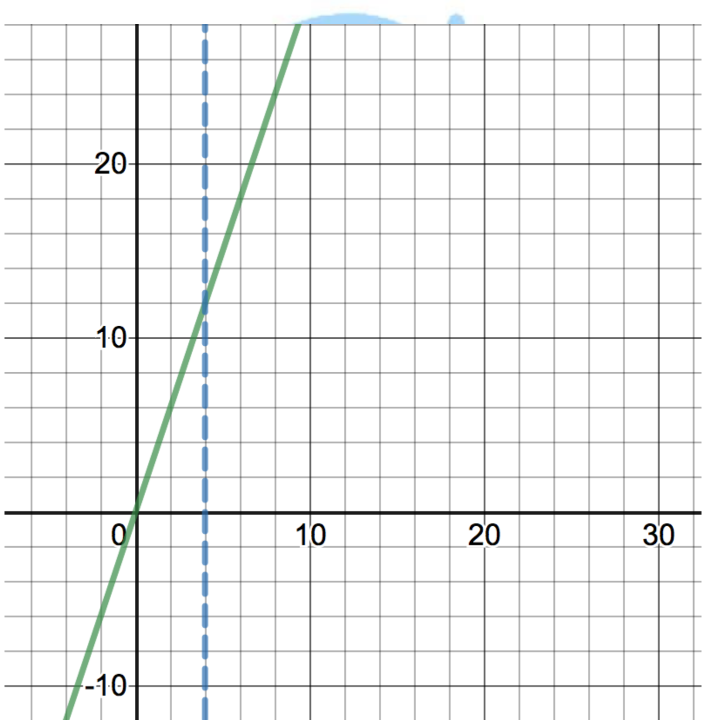
Limits

Limits can be used in calculus to determine the continuity of a function.

Given a function $y = f(x)$, the notation $f(x)$ is used to find the numerical y-value that the function approaches as the x-value gets infinitely closer to c.

Finding Limits Graphically

Given the function y = 3x, find the limit as x approaches 4, also known as $3x$:



Look at the y-value as you approach the x-value of 4 from the left and as you approach the x-value of 4 from the right. If the y-values are the same, then the limit exists and is equal to the y-value. If the y-values are not the same, then the limit does not exist.

Finding Limits Analytically

⇒ <u>Direct Substitution –</u> this method is for simpler limits in which you can substitute the x-value that the limit is approaching into the f(x) equation to find the limit.

- Example: $5x - 3$ □ $5(2) - 3 = 7$

⇒ <u>Factoring & Canceling –</u> when the function that you're finding a limit for is a rational function, factor out the numerator and denominator of the function, cancel out the factors(s) that appear on both the numerator and the denominator, and then use direct substitution on the resulting equation.

- Example: $\frac{x^3 - 1}{x^2 - 1}$ □ $\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1} = \frac{(1)^2+(1)+1}{(1)+1} = \frac{3}{2}$

⇒ <u>Rationalizing the Numerator –</u> rationalizing is usually done to eliminate the root from the denominator of a fraction, but in the case of finding limits it becomes necessary to rationalize the numerator. This can be done by multiplying both the numerator and denominator by the numerator's conjugate, distributing only the numerator, canceling out the factor that is on both the numerator and the denominator, and then using direct substitution.

- Example: $\frac{\sqrt{x}-\sqrt{3}}{x-3}$ □ $\frac{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}{(x-3)(\sqrt{x}+\sqrt{3})} = \frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})} = \frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$

⇒ <u>Special Trigonometric Cases –</u> functions that contain certain trigonometric functions have special cases for finding limits that need to be memorized. Note that these cases only apply when the limit to be found occurs when the function is approaching the x-value zero.

- $\frac{\sin x}{x} = 1$; $\frac{x}{\sin x} = 1$
  - As long as the component inside the sine function is the same as the denominator/numerator, the special case applies.
- $\frac{1 - \cos x}{x} = 0$; $\frac{\cos x - 1}{x} = 0$
  - As with the case before this, if the component inside the cosine function is the same as the denominator, the special case applies.

Continuity

- ⇒ A function is continuous on an open interval (a, b) if there are no asymptotes, holes, and/or jumps at any point on that interval. If it is possible to graph the function without lifting your pencil from the paper, then the function is considered to be continuous.
- ⇒ If a function is discontinuous at a point *c*, then (c, f(c)) is considered a point of discontinuity.
- ⇒ A point of discontinuity of a function *f* is called <u>removable</u> *f* can be made continuous by redefining *f*, by factoring and canceling. This type of discontinuity is called a hole in the graph.
- ⇒ If it is impossible to redefine *f*, then the point of discontinuity is <u>non removable</u>. This type of discontinuity is called a vertical asymptote.

One-Sided Limits

- ⇒ $f(x)$ means “the limit as x approaches c from the right”
- ⇒ $f(x)$ means “the limit as x approaches c from the left”

Intermediate Value Theorem

- ⇒ If *f* is continuous on the closed interval [a, b] and *k* is any y-value between f(a) and f(b), then there is at least one number *c* on the interval [a, b] such that f(c) = *k*.
- ⇒ This theorem can be used when you are given the function’s equation, the interval [a, b], the value of *k*, and you are asked to find the value of *c*.
    - First, you would find the start y-value and the end y-value by plugging in the start and . end x-values from the closed interval into the given formula. These values will be used to ensure that the *k* value — or f(c) — occurs inside the interval.
    - To find *c*, substitute the variable *c* in for the variable *x* into the equation of the function, substitute the *k* value for f(x), and solve for *c*. Make sure that the value(s) for *c* that you come up with are within the domain restrictions given by the closed interval.

Infinite Limits

- ⇒ If the limit you're trying to find occurs at a point where the function has an asymptote, this is called an infinite limit.
- ⇒ Infinite limits occur when the limit (L) approaches ∞ or -∞, based on the direction of the graph.
- ⇒ Note that ∞ and -∞ are abstract regions that can never be reached and are therefore not real numerical y-values and thus, we say that the limit does not exist.

Differentiation

- ⇒ The instantaneous slope of a function *f* at any point can be calculated by the following equation:
    - $m = \frac{f(x+\Delta x) - f(x)}{\Delta x}$
    - This is the slope of a tangent line at any point on function *f*
- ⇒ The instantaneous slope is also referred to as the derivative of *f*, and the process of calculating the derivative is called differentiation.
- ⇒ Special notation that represents the derivative of *f*:
    - f'(x) ~ "f prime of x"
    - $\frac{dy}{dx}$ ~ "the derivative of y with respect to x"
    - y' ~ "y prime"
    - $\frac{d}{dx}[f(x)]$ ~ "the derivative of f(x) with respect to x"
- ⇒ A function is not differentiable at any point of discontinuity, anywhere there is a sharp point in an absolute value graph, at any point where the slope is vertical, or at any endpoint.

Differentiation Rules

- ⇒ Constant Rule – if f(x) is a constant function without a variable, then f'(x) = 0.
- ⇒ If it is a linear function, the derivative is the coefficient of the *x* variable.
- ⇒ Power Rule – if f(x) = x^n where *n* is a rational number, then f'(x) = $nx^{(n-1)}$.
    - Example: f(x) = x^4 □ f'(x) = $4x^3$

⇒ <u>Constant Multiple Rule –</u> If *f* is differentiable and *c* is a real number, the *cf* is also differentiable and $\frac{d}{dx}[cf(x)]$ = cf'(x).

- Example: f(x) = $-2x^3$ □ f'(x) = –2 × $3x^2$ = $-6x^2$

⇒ <u>Sum and Difference Rules –</u> if the function as multiple terms that are being added and/or subtracted together, then the derivative of the function is the derivative of each of the terms added and/or subtracted together.

- Example: f(x) = $-2x^3$ + $3x^2$ – 4x □ f''(x) = $-6x^2$ + 6x – 4

⇒ <u>Product Rule –</u> if *f* and *g* are differentiable, $\frac{d}{dx}f(x)g(x)$ = f(x)g'(x) + g(x)f'(x)

- Translation: take the first factor multiplied by the derivative of the second factor and add it to the second factor multiplied by the derivative of the first factor.

⇒ <u>Quotient Rule –</u> if *f* and *g* are differentiable, $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)f'(x) - f(x)g\prime(x)}{[g(x)]^2}$

- Translation: take the component in the denominator multiplied by the derivative of the component in the numerator and subtract from it the component in the numerator multiplied by the derivative of the component of the denominator, and put all of that over the component in the original denominator squared.
- Easy way to remember: lo di hi minus hi di lo all over lo lo.

⇒ <u>Chain Rule –</u> this rule is used for composite functions, and parentheses are usually indicators that you should use this rule. If *f* and *g* are differentiable, $\frac{d}{dx}f(g(x))$ = f'(g(x)) × g'(x)

- Translation: take the derivative of the outside function – leaving the inside function as it is – and multiply by the derivative of the inside function.
- This rule is used for composite functions, and parentheses are usually indicators that you should use this rule.

⇒ <u>Trigonometric Rules –</u>

- Sine ~ $\frac{d}{dx}sin\ x$ = cos x
- Cosine ~ $\frac{d}{dx}cos\ x$ = –sin x
- Tangent ~ $\frac{d}{dx}tan\ x = sec^2\ x$
- Secant ~ $\frac{d}{dx}sec\ x$ = (sec x)(tan x)

- Cosecant ~ $\frac{d}{dx} csc\ x$ = (csc x)(cot x)
- Cotangent ~ $\frac{d}{dx} cot\ x = csc^2\ x$

Higher Order Derivatives

1. <u>Position Function –</u> s(t) = $-16t^2$ + 98t + 407
2. <u>Instantaneous Velocity Function –</u> s'(t) = –32t + 98 ~ first derivative
3. <u>Acceleration Function –</u> s''(t) = a(t) = –32 ~ second derivative

Implicit Differentiation

⇒ Used for differentiating a component with a different variable with respect to x – for the purpose of this section, we'll focus on the y-variable, but implicit differentiation can work for more than one other variable in the function.

⇒ To do implicit differentiation, take the derivative of the terms with y-variables regularly the same rules as you would for terms with x-variables, and for each term with y-variables that you take the derivative of you multiply them by a y'.

- Example: $\frac{d}{dx} y^3 = 3y^2$ × y' = $3y^2y'$

Related Rates

⇒ This process can be used to find the rate of change for something that is said to have a relationship with a given rate of change.

⇒ The rates of change are usually in respect to time.

⇒ A lot of times this process is employed when talking about shapes that have proportions that are changing.

⇒ Steps:

- Write the formula (of the geometric shape).
- Rewrite the formula as needed so it only has two variables that are changing.
- Differentiate with respect to time (t).
- Substitute in the given information and simplify. Remember that rates of change are the same as derivatives.

- ⇒ Example: the side of a square is changing at a rate of 0.2 square feet per second. Find the rate the area of the square is changing at the instant the side is 16 feet.
  - Formula – A = s^2
  - It's not necessary to rewrite this formula.
  - $\frac{dA}{dt} \times 1 = 2s \times \frac{ds}{dt}$
  - $\frac{dA}{dt} = 2(16) \times (0.2 \text{ ft/sec})$ □ $\frac{dA}{dt} = 6.4\ ft^2/sec$

Applications of Differentiation

Extrema on an Interval

- ⇒ Absolute Extrema – extreme values of the function, either the minimum (lowest) or the maximum (highest) points of a function.
- ⇒ Extrema occur at turning points or endpoints, and they are determined by the y-value.
- ⇒ Extrema Value Theorem – if *f* is continuous on a closed interval [a, b], then *f* has both a minimum and maximum on the interval.
- ⇒ Relative Extrema – they occur at turning points. They are either the lowest one of the points to the immediate left or right (relative minimum), or the highest of one of the points to the immediate left or right (relative maximum).
  - Relative extrema will occur at turning points where the first derivative is zero (a horizontal tangent) or where the first derivative is undefined (a sharp point).
  - The x-values at these special points are called critical numbers. Note that if the f'(c) = 0, the *c* is a critical number, but is not a relative extrema.

Rolle's Theorem

- ⇒ The function *f* is continuous on the closed interval of [a, b] and differentiable on the open interval (a, b).
- ⇒ If f(a) = f(b) – the y-values of the endpoints are equal – then there is at least one number *c* in (a, b) such that f'(c) = 0.
- ⇒ Steps:

- First verify that Rolle's Theorem can be used by substituting the x-values of the open interval into the function's equation to make sure that their y-values are equal.
- Substitute the variable *c* in for x.
- Solve for *c*.
- Eliminate any values for *c* that aren't inside the open interval.

Mean Value Theorem

⇒ The function *f* is continuous on the closed interval of [a, b] and differentiable on the open interval (a, b).

⇒ There exists a number *c* in the interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. This means that the tangent line at the point *c* will have the same slope as a line connecting the two endpoints, meaning that they will be parallel.

⇒ Steps:

- Find the y-values of both the endpoints.
- Take the derivative of the function and substitute *c* in for the variable x so that the derivative becomes f'(c).
- Plug the endpoints into the slope formula and set it equal to f'(c).
- Solve for *c*.
- Eliminate any values for *c* that aren't inside the given interval.

Increasing and Decreasing Intervals

⇒ In general, the slope of the tangent (derivative) is negative when the function is decreasing.

⇒ In general, the slope of the tangent line is positive when the function is increasing.

⇒ In general, the slope of the tangent is zero when the function is constant (neither increasing nor decreasing).

⇒ Steps to finding increasing and decreasing intervals of a function analytically:

- Find the function's critical numbers – i.e. the x-values where the first derivative is either 0 or undefined. Remember that critical numbers are sometimes considered to be relative extrema.
- Use the critical numbers and any vertical asymptotes to create a number line for test intervals.
- Pick one number inside each test interval and plug it into the first derivative.
- The sign of the resulting first derivatives will determine what the function is doing in the test intervals. If f'(x) is negative, the function is decreasing; if f'(x) is positive, the function is increasing.

The First Derivative Test

⇒ If f'(x) changes from positive to negative at a critical number *c*, then *c* is a relative maximum.

⇒ If f'(x) changes from negative to positive at *c*, then *c* is a relative minimum.

⇒ If f'(x) does not change signs at *c*, then *c* is neither a relative maximum nor a relative minimum.

Concavity

⇒ The concavity of a function relates to which way the function points of "bend," and the concavity is related to the first derivative, as the first derivative dictates whether the function is increasing or decreasing.

⇒ If the function *f* is increasing over an interval, then the function is concave up.

⇒ If *f* is decreasing over an interval, then the function is concave down.

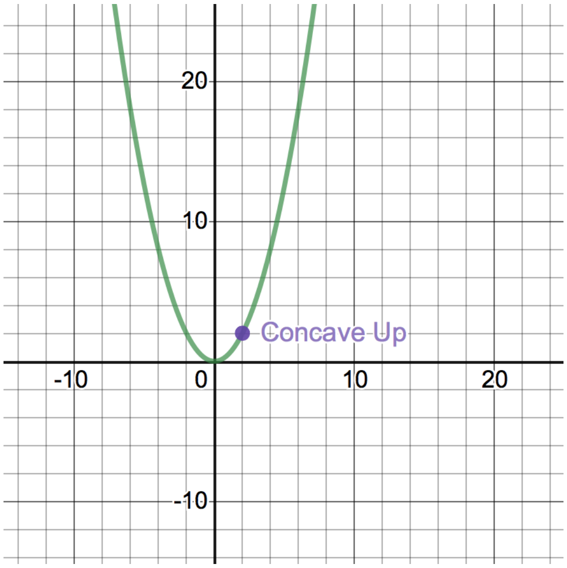
⇒ The concavity of a function can change at a point called a point of inflection.

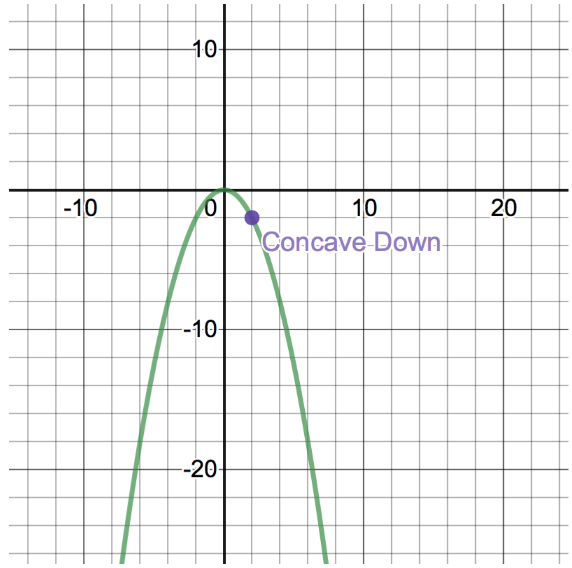
⇒ If f''(c) is positive, then *f* is concave up at *c*.

⇒ If f''(c) is negative, then *f* is concave down at *c*.

⇒ If f''(c) is zero, then *c* is a point of inflection.

⇒ The same steps for finding increasing/decreasing intervals can be used to figure out concavity of a function, except instead of using the first derivative you use the second derivative.





The Second Derivative Test

⇒ Steps to find relative extrema using the second derivative test:
  - Take the first derivative of the function.
  - Set the first derivative equal to zero and solve for x in order to find the function's critical numbers.
  - Take the derivative of the first derivative to create the second derivative.
  - Substitute each critical number into the second derivative and determine the sign of the second derivative at each critical number.

⇒ If f''(c) is positive, then f(c) is a relative minimum.

⇒ If f''(c) is negative, then f(c) is a relative maximum.

⇒ If f''(c) is equal to zero, then the test fails.

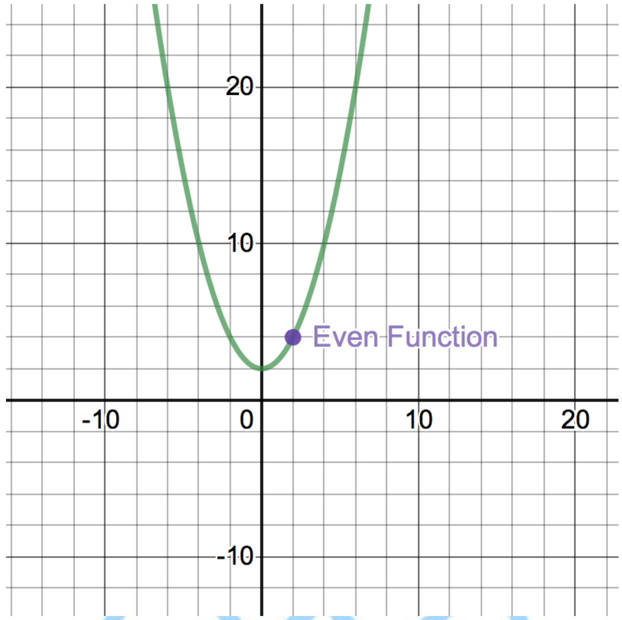
Limits at Infinity

⇒ This concerns when you're trying to find limits where the x-value is approaching positive or negative infinity as opposed to a real numerical value.

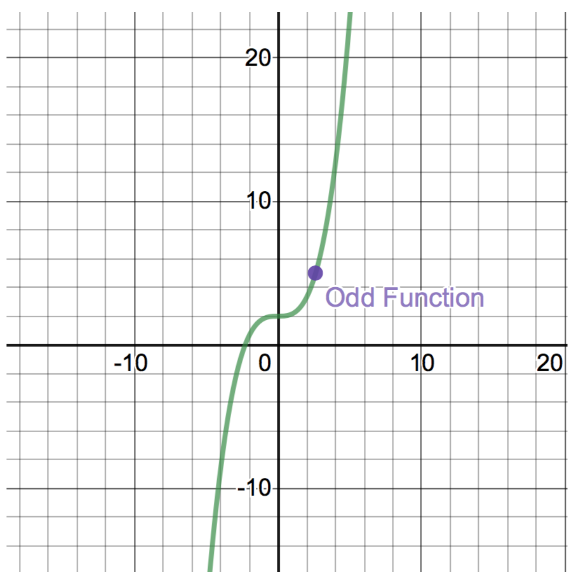
⇒ Most problems with limits at infinity concern rational functions.

⇒ Limits at infinity result in the function having horizontal asymptotes.

⇒ If the exponent on the x-variable of the function is even, then the function will have a graph where both sides go towards positive infinity, regardless of whether the limit approaches positive or negative infinity.



⇒ If the exponent on the x-variable of the function is odd, then the function will have a graph where the two sides go towards opposite directions – when the limit approaches positive infinity, the graph will go towards positive infinity, but when the limit approaches negative infinity, the graph will go towards negative infinity.



- ⇒ If in a rational function the x-variable is on the denominator and there is only a constant in the numerator, then the limit equals zero.
- ⇒ If in a rational function the exponent of the x-variable in the denominator is the same as the exponent of the x-variable in the numerator, then the limit is equal to the constant in front of the x-variable.
- ⇒ Steps for finding limits at infinity for rational functions:
  - Divide both the numerator and the denominator by the variable with the highest exponent in the denominator.
  - Find the limit of each term individually.
  - Simplify the fraction.
- ⇒ Example: $\frac{2x-1}{x+1}$
  - $\frac{\frac{2x}{x}-\frac{1}{x}}{\frac{x}{x}+\frac{1}{x}} = \frac{2-0}{1+0} = 2$ ~ horizontal asymptote at y = 2
- ⇒ If the exponent in the numerator is exactly one power higher than the exponent in the denominator, then the function has an oblique or slant asymptote. To solve oblique/slant asymptote, you can find the limit at infinity by doing long division.

Optimization

- ⇒ The purpose of optimization is to find the smallest or largest value of a function given the dimensions of the other components of the function.
- ⇒ Many optimization problems lean on real-world applications, so many of the questions pertain to geometric shapes such as cubes or rectangular prisms – rectangular prisms are popular to use for these problems, because it relates to how companies try to optimize space in boxes.
- ⇒ With optimization problems, it helps to draw/sketch what the problem is asking you to optimize and label the sketch with the coinciding dimensions given, so that you can better visualize the problem.
- ⇒ Optimization problems will have a primary equation that needs to be optimized and a secondary equation that is known based on the information in the problem.
- ⇒ Steps for optimization:

- Establish the primary equation.
- Establish the secondary equation. You may need to change the secondary equation accordingly to have the same variables as the primary equation.
- Solve for one of the variables in the secondary equation.
- Substitute the resulting secondary equation into the primary equation so that the primary equation has two variables and simplify.
- Take the derivative of the new primary equation and get the critical numbers. The positive critical number will be one of the dimensions.
- Solve for the remaining variable that was isolated in the third step.

⇒ Example: a manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

- Primary equation ~ V = l^2h
- Secondary equation ~ SA = 108 □ l^2 + 4lh = 108
- Solve for h ~ 4lh = 108 – l^2 □ h = $\frac{108 - l^2}{4l}$
- Substitution ~ V = $l^2 \times \frac{108 - l^2}{4l} = \frac{108l^2 - l^4}{4l} = \frac{180l^2}{4l} - \frac{l^4}{4l} = 27l - \frac{1}{4}l^3$
- Derivative ~ V' = $27 - \frac{3}{4}l^2$; Critical numbers ~ $l = \pm 6$ □ $l = 6$
- Solve for h ~ h = $\frac{108 - (36)}{4(6)} = 3$
- Dimensions ~ length – 6 in; width – 6 in; height – 3 in

Integration

Antiderivatives and Indefinite Integrals

⇒ Antidifferentiation is the inverse operation of differentiation. A good way to check that you've done an antidifferentiation problem is to take the derivative of the answer you get – if the derivative of the answer is the same as the original question, then you did the problem correctly.

⇒ The symbol that is used to indicate that you'll need to take the antiderivative of f(x) is $\int \quad f(x) \times dx.$

- This is read as the integral of f(x) with respect to x.

⇒ Integration is a synonym for antidifferentiation.

⇒ When taking an indefinite integral, remember that the derivation process eliminates constants, because the derivatives of constants are equal to zero, according to the constant rule. Because of this, indefinite integrals require that you write a “+ C” at the end of your resulting equation, to account for the unknown constant.

⇒ Power Rule of integration – add one to the exponent, and then divide the term by the new exponent.

⇒ You must always remember the special trig derivative rules.

Summations

⇒ Summations are used to add up all the components in a set.

⇒ The notation for summation is $\sum_{i=1}^{n} i$, where

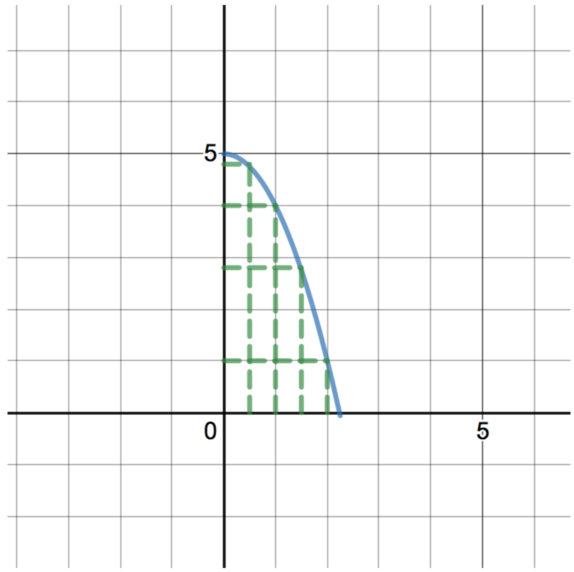
- n is the end of the set.
- *“i =”* indicates the start of the set.
- i is the formula for the variables in the set.

⇒ Rules for summation:

- Constant rule – $\sum_{i=1}^{n} c = c \times n$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$
- $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4+2n^3+n^2}{4}$

Estimating Area Under a Curve

Consider a graph of y = $-x^2 + 5$ on a closed interval of [0, 2] with 4 inscribed – or right-bound – rectangles.



Summations can be used to calculate an estimated value of the area under the graph's curve.

- ⇒ In general, the width of one rectangle is $\Delta x = \frac{b-a}{n}$.
  - ○ a is the lower bound of the interval.
  - ○ b is the upper bound of the interval.
  - ○ n is the number of rectangles.
- ⇒ If the rectangle's height is determined by the right side of the rectangle, then a way to describe the x-coordinate of the right end of the triangle can be $(\frac{b-a}{n})$i.
  - ○ i is the rectangle that you're looking at.
- ⇒ Therefore, in general, the height of each rectangle can be represented by f(($\frac{b-a}{n}$)i).
- ⇒ Using summation, the area under a curve with inscribed rectangles can be found by using the formula $\sum_{i=1}^{n} \quad f(\left(\frac{b-a}{n}\right) i) \times \frac{b-a}{n}$.
- ⇒ With circumscribed, or left-bound, rectangles the height changes to f(($\frac{b-a}{n}$)i).
- ⇒ Using summation, the area under a curve with circumscribed rectangles can be found using the formula $\sum_{i=1}^{n} \quad f(\left(\frac{b-a}{n}\right)(i-1)) \times \frac{b-a}{n}$.

Definite Integrals

- ⇒ Unlike indefinite integrals, definite integrals pertain to specific bounded regions.
- ⇒ The notation for definite integrals is $\int_a^b f(x)dx$.
  - a is the lower bound.
  - b is the upper bound.
- ⇒ Because this type of integral has definite bounds, there is no need for the addition of a "+ C" after integration. Instead, after the integration process the equation will look like this:
  - F(x) $\Big|_a^b$, where F(x) is the antiderivative.
  - The vertical integration bar means that you substitute the bounds in for x.
  - After substituting in the bounds, subtract F(a) from F(b) to get the answer to the definite integral.
  - Remember that in this process the upper bound goes first.

Rules of Integration

- ⇒ An integral must be continuous and differentiable on a closed interval [a, b].
- ⇒ If the upper and lower bounds are the same, the integral's value is zero.
- ⇒ $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- ⇒ If $a < c < b$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- ⇒ If k is a constant, then $\int_a^b k \times f(x)dx = \mathrm{k} \times \int_a^b f(x)dx$
- ⇒ $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Average Value of a Function

- ⇒ If f(x) is continuous on a closed interval [a, b] then at a point c the function will reach its average y-value, denoted as f(c).
- ⇒ The average value of a function can be found using f(c) = $\frac{1}{(b-a)} \times \int_a^b f(x)dx$.

Fundamental Theorem of Calculus

- ⇒ The Fundamental Theorem of Calculus is, essentially, the derivative of a definite integral.
- ⇒ The Fundamental Theorem of Calculus works as a way to establish a relationship between differentiation and integration.
- ⇒ The Fundamental Theorem of Calculus also guarantees that any continuous function has an antiderivative.
- ⇒ One of the most important implications of the Fundamental Theorem of Calculus is that if one is able to find the antiderivative of an integrand – which is an equation that needs to be integrated – then the definite integral can be evaluated by evaluating the antiderivative at the interval's endpoints and subtracting them.
- ⇒ The equation for the Fundamental Theorem of Calculus is:
    - $\frac{d}{dx}\int_{a}^{x} f(t)dt = f(x)$
    - a is a constant
    - Translation: substitute the upper bound in and multiply that quantity by the derivative of the upper bound, and subtract from that the substitution of the lower bound multiplied by the derivative of the lower bound.
    - Since the upper bound is the variable x and the lower bound is a constant, this will result in F(x) – F(a).
    - Because F(a) will come out as a constant number and the derivative of a constant is zero – according to the constant rule of differentiation – the result of this theorem will solely be F'(x), or f(x).

Integration by Substitution

- ⇒ Substitution – also called u-sub – is basically the antiderivative of the chain rule.
- ⇒ It is used to integrate composite functions by substituting in the variable *u* in place of the composite during the integration process.
- ⇒ Usually, *u* is substituted in for the inner function of the composite.
- ⇒ Steps:
    - Determine whether the function is composite, and which part of the function needs to use substitution.

- Take the determined part of the composite and set it equal to u.
- Take the derivative of this u equation so that on one side you have $\frac{du}{dx}$.
- Solve for dx. This will be used to substitute into the original integration problem.
- Go back to the original integration problem and substitute u in for the composite function and the dx equation found in the previous step for dx.
- Simplify inside the integrand before integrating so that the x-variables cancel.
- Integrate.
- Substitute the composite function back in for u after integration.

⇒ Example: $\int (x^2+1)^2(2x)dx$

- Composite function to use substitution for – x^2 + 1
- u = x^2 + 1
- $\frac{du}{dx}$ = 2x
- du = 2x × dx □ dx = $\frac{du}{2x}$
- $\int u^2 \times 2x \times \frac{du}{2x}$
- $\int u^2 \times du$
- $\frac{u^3}{3}$ + C
- $\frac{(x^2+1)^3}{3}$ + C

⇒ If after substituting in u and the equation for dx and simplifying the integrand still has an x-variable, go back to when you set the composite function equal to u and solve for x in terms of u. Plug the resulting function in for x in the integrand.

Logarithmic, Exponential, and Inverse Functions

Natural Logarithmic Functions – Properties

⇒ Recall that the natural log is written as "$log_e x$" or "ln x." Therefore, a natural logarithmic function is written s "f(x) = ln x."

⇒ In x = $log_e x$, e ≈ 2.72

⇒ ln(1) = 0

⇒ ln(ab) = ln a + ln b

⇒ $\ln(\frac{a}{b})$ = ln a – ln b

⇒ $\ln(a^n)$ = n × ln a

⇒ $log_m\, n = \frac{log\, n}{log\, m} = \frac{ln\, n}{ln\, m}$

Natural Logarithmic Functions – Differentiation

⇒ $\frac{d}{dx}[ln\, x] = \frac{1}{x}$, x > 0

⇒ If there is a function inside the natural log other than x, the chain rule applies.

- $\frac{d}{dx}[ln\, u] = \frac{1}{u} \times u'$ <u>or</u> $\frac{d}{dx}[ln\, u] = \frac{u'}{u}$

⇒ Example:

- f(x) = ln(2x)
- f'(x) = $\frac{1}{2x} \times 2$
- f'(x) = $\frac{1}{x}$

⇒ $\frac{d}{dx}[ln\, |u|] = \frac{u'}{u}$

Natural Logarithmic Functions – Integration

⇒ Recall, $\frac{d}{dx}[ln\, u] = \frac{u'}{u}$

⇒ Therefore, $\int \frac{u'}{u} \times du$ = ln |u| + C and $\int \frac{1}{x} \times dx$ = ln |x| + C

⇒ When integrating natural logs, there is a new choice for *u* in u-sub.

⇒ If the power of the denominator is one higher than the power of the numerator, the denominator can be set equal to *u* for substitution.

⇒ If the power of the numerator is greater than or equal to the denominator, then you have to do long division first before integrating.

Inverse Functions

- ⇒ *f* and *g* are inverse functions if, and only if, f(g(x)) = x for each x-value in the domain of *g* and g(f(x)) = x for each x-value in the domain of *f*. This means that when you put in any y-value of *g* into *f*, the y-value of the function *f* that will be produced is the corresponding x-value in the function *g*, and vice versa.

⇒ Inverse functions are the "reversals" of functions.

⇒ *f* and *g* are inverse functions if, and only if, f(g(x)) = x for each x-value in the domain of *g* and g(f(x)) = x for each x-value in the domain of *f*. This means that when you put in any y-value of *g* into *f*, the y-value of the function *f* that will be produced is the corresponding x-value in the function *g*, and vice versa.

⇒ If *g* is the inverse of *f*, then *g* can be denoted as f^{-}.

⇒ Graphically, the inverses are reflected over the line y = x and the ordered pairs are switched.

⇒ Ways to tell if a function has an inverse:

- Horizontal line test – if a horizontal line can be drawn that intersects the function in more than one place, then the function does NOT have an inverse. If it passes the horizontal line test, then the function is said to be one-to-one.
- Strictly monotonic – a function is said to be strictly monotonic if it always only increases or decreases over its entire domain; this means that the function will have no extrema.

Exponential Functions – Properties

⇒ f(x) = ln x and g(x) = e^x are inverse functions.

⇒ $\ln(e^x) = x$

⇒ $e^{(\ln x)} = x$

⇒ $e^{(a+b)} = e^a \times e^b$

⇒ $e^{(a-b)} = \frac{e^a}{e^b}$

⇒ $e^{ab} = (e^a)^b$

- Example: $e^{2x} = (e^x)^2$

Exponential Function – Differentiation

⇒ If f(x) = e^x, then f'(x) = e^x

⇒ If the exponent is a function other than just x, use the chain rule.

- If f(x) = e^u, then f'(x) = $e^u \times u'$

⇒ Example: f(x) = $e^{(3x-2)}$ □ f'(x) = $3e^{(3x-2)}$

Exponential Functions – Integration

⇒ Since $\frac{d}{dx} e^x = e^x$, $\int e^x \times dx = e^x + C$

⇒ With integrating exponential functions, there is now a third option for when to use u-substitution: in the exponent of the exponential function, if the exponent is not just x.

- Example: $\int e^{(3x-2)} \times dx$
- u = 3x – 2
- $\frac{du}{dx}$ = 3 □ dx = $\frac{du}{3}$
- $\int e^u \times \frac{du}{3}$
- $\int \frac{1}{3} \times e^u \times du$
- $\frac{1}{3} e^u$ + C
- $\frac{1}{3} e^{(3x-2)}$ + C

Logarithms with Bases Other Than e – Differentiation

⇒ Recall that ln x = $log_e\, x$.

⇒ If a log has a base of a, a constant, then the derivative is y' = $\frac{1}{x} \times \frac{1}{\ln a}$.

⇒ If a log with a base of a has something inside the log function other than x, the derivative becomes y' = $\frac{1}{u} \times$ u' $\times \frac{1}{\ln a}$ <u>or</u> y' = $\frac{u'}{u} \times \frac{1}{\ln a}$.

⇒ Explanation:

- y = $log_a\, x$
- Recall the property of natural logs that states: $log_m\, n = \frac{\log n}{\log m} = \frac{\ln n}{\ln m}$
- y = $\frac{\ln x}{\ln a}$
- y = $\frac{1}{\ln a} \times$ ln x
- y' = $\frac{1}{x} \times \frac{1}{\ln a}$

Exponentials with Bases Other Than e – Differentiation

- ⇒ If an exponential has a base of a number a, then the derivative is y' = $a^x(ln\ a)$.
- ⇒ If the exponential with a base of *a* has something inside the exponent other than x, then the derivative is y' = a^u × u' × ln a
- ⇒ Explanation: this formula can be created by using logarithmic differentiation, which is the process of taking the ln of both sides, using ln properties, and then differentiating.
  - y = a^x
  - ln y = ln a^x
  - ln y = x ln a
  - $\frac{1}{y}$ × y' = 1 × ln a
  - y' = y × ln a
  - Remember that y = a^x and substitute that in for y.
  - y' = $a^x(ln\ a)$

Bases Other Than e – Integration

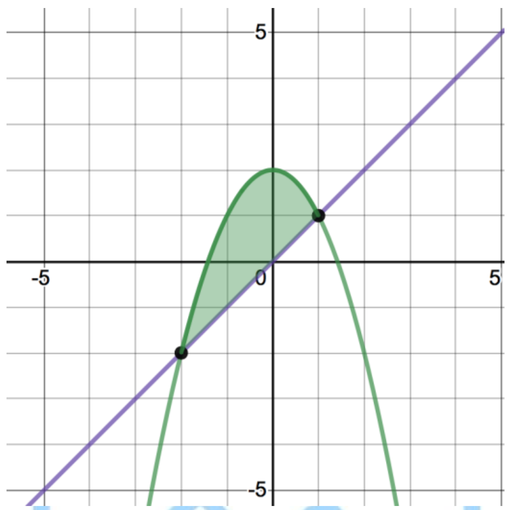
- ⇒ $\int \quad a^x \times dx = \frac{a^x}{ln\ a}$ + C
- ⇒ Don't forget to use u-substitution if the exponent is something other than just x.
- ⇒ Example: $\int \quad 5^{4x^2} \times x \times dx$
  - u = $4x^2$
  - $\frac{du}{dx}$ = 8x □ dx = $\frac{du}{8x}$
  - $\int \quad 5^u \times x \times \frac{du}{8x}$
  - $\int \quad \frac{1}{8} \times 5^u \times du$
  - $\frac{1}{8} \times \frac{5^u}{ln\ 5}$ + C
  - $\frac{5^{4x^2}}{8\ ln\ 5}$ + C

Applications of Integration

Area of a Region Between Two Curves

⇒ If f(x) > g(x) for all x-values in a closed interval [a, b], the area of the gap between the two curves f(x) and g(x) can be found by: A = $\int_a^b [f(x) - g(x)]dx$

⇒ Graphically, this means that the first function, denoted as f(x), must be “above” the second function g(x) over the whole interval.



⇒ Example: find the area of the region bounded by f(x) = 2 – x^2 and g(x) = x – the graph of these two functions and the region bounded by them is pictured above.

- Finding the points of intersection (to know the bounds) –
  - 2 – x^2 = x
  - x^2 + x – 2 = 0
  - (x + 2)(x – 1) = 0
  - x = –2; x = 1
- A = $\int_{-2}^{1} [(2 - x^2) - x]dx$
- A = 2x – $\frac{1}{3}x^3$ – $\frac{1}{2}x^2$ $\Big|1 \ -2$
- A = $\frac{9}{2}$ $units^2$

Volume Using the Disk Method

⇒ When a function is revolved over the x-axis, the volume of the solid figure that results from the function's revolution can be found through something called the Disk Method.

⇒ The formula for this method is: V = $\pi \int_a^b [f(x)]^2 dx$ <u>or</u> V = $\pi \int_a^b [R(x)]^2 dx$.

- f(x) = R(x), where R is the radius of the disk. This is the representative rectangle.

⇒ The Disk Method is related to the formula for the volume of a cylinder, $V_{cyl} = \pi r^2 h$.

⇒ Example: find the volume of the solid generated by revolving the region bounded by the function f(x) = $\sqrt{\sin x}$ and the x-axis for $0 \leq x \leq \pi$ around the x-axis.

- V = $\pi \int_0^{\pi} (\sqrt{\sin x})^2 dx$
- V = $\pi \int_0^{\pi} \sin x \times dx$
- V = $\pi \times$ [-cos x – (-cos 0)]
- V = 2 π

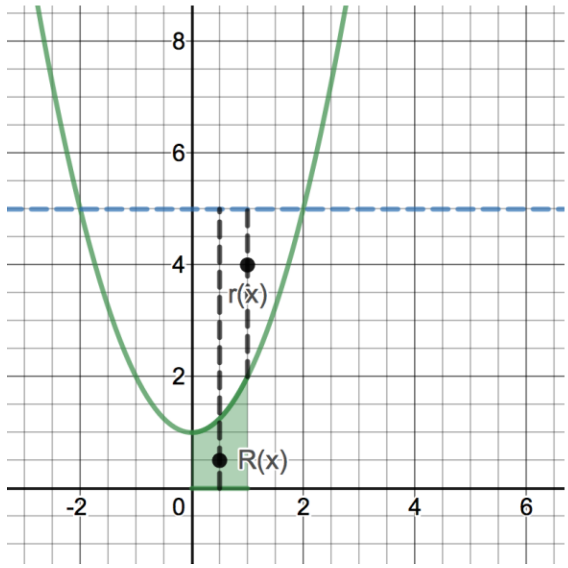
Volume Using the Washer Method

⇒ The Washer Method is used if the solid that is formed after revolving a function around the x-axis (or any line) has a "hollow" region, where part or all of the function does not intersect the line that the function is revolved around.

⇒ An easy way to discern whether to use this method is if the region being rotated is bounded by two curves/lines that are different than the line that the region is being revolved around.

⇒ The formula for the Washer Method is: V = $\pi \int_a^b \{[R(x)]^2 - [r(x)]^2\} dx$

- Using the rules of integration, the formula is also:
  - V = $\pi \int_a^b [R(x)]^2 dx - \pi \int_a^b [r(x)]^2 dx$
- R(x) is the larger radius and r(x) is the smaller radius from the axis of revolution.
- This means that R(x) is the vertical distance – meaning only looking at the y-values – between the line of revolution and the top of the region you're finding the volume for, and r(x) is the vertical distance between the line of revolution and the top of the "hollow" region.



⇒ Example: find the volume of the solid formed by revolving the region bounded by the graphs of y = x^2 + 1, y = 0, x = 0, and x = 1 about the line y = 5 (graph above):

- R = 5 – 0 = 5
- r = 5 – (x^2 + 1) = 4 – x^2
- $V = \pi \int_0^1 \; [(5)^2 - \left(4 - x^2\right)^2] dx$
- $V = \pi \int_0^1 \; \left(25 - 16 + 8x^2 - x^4\right) dx$
- $V = \pi \int_0^1 \; \left(9 + 8x^2 - x^4\right) dx$
- $V = \pi \times [9x + \frac{8}{3}x^3 - \frac{1}{5}x^5 \; |1 \;\; 0]$
- $V = \pi \times (9 + \frac{8}{3} - \frac{1}{5})$
- $V = \frac{172\pi}{15}$