Math AA HL Year 1 Guide

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Topic 1: Number and Algebra

PART A

Number systems:

R → Real Numbers

The set of real numbers includes all the rational and irrational numbers. The real numbers are "all the numbers" on the number line. There are infinitely many real numbers just as there are infinitely many numbers in each of the other sets of numbers.

Q → Rational Numbers

Rational Numbers are numbers that can be expressed as a ratio of two integers, or by the fraction p/q, where p and q are integers. For instance, the fractions $\frac{1}{2}$ and $\frac{-7}{23}$ are rational numbers. Another very important thing to note here is that only terminating decimals are rational numbers. Decimals that follow a repeating pattern after some point are also rationals: for example, 0.333333 = $\frac{1}{3}$ is a rational number.

Z → Integers

Integers are the set of real numbers consisting of natural numbers or whole numbers, their additive inverses and zero. The sum, product and difference of two integers is also an integer. For example: {1,2,32,-4,20} are all integers.

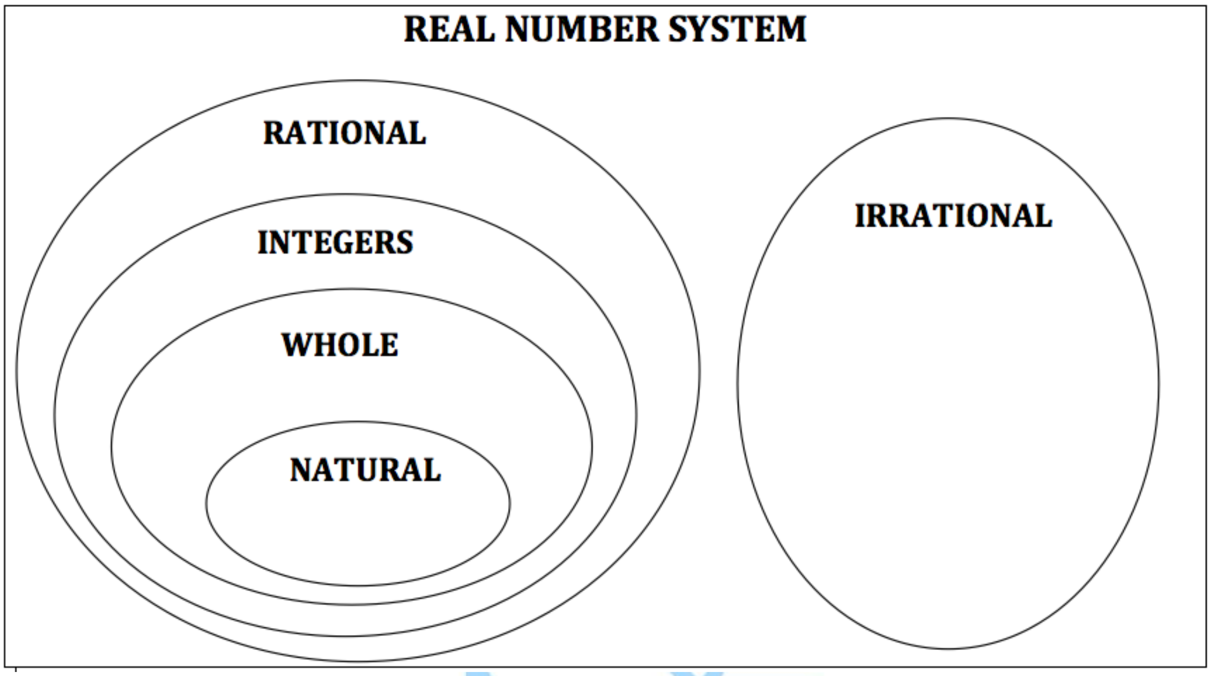
N → Natural Numbers

The natural numbers are essentially counting numbers like 1,2,3,4,5, etc. There are infinitely many natural numbers.

I → Irrational Numbers

An irrational number is a number that cannot be written as a ratio of two integers. In its decimal form, it never ends or repeats. For instance, $\sqrt{3}$ and $\pi = 3.14159265358979..$, are both irrational numbers.

The following venn diagram displays the conventional number systems that are involved in mathematics: Real Numbers, Rational Numbers, Integers, Natural Numbers and Irrational Numbers. It gives an understanding of how these number systems are related and identified.



PART B

Arithmetic Sequences and Series:

Finding an equation of an arithmetic sequence is essentially like finding a pattern in the numbers. A sequence is a list of numbers that is written in a defined order which can be ascending or descending. Each of the numbers in a sequence is called a term, and an arithmetic sequence is often called a **progression.**

A sequence is defined with a general formula shown below.

The nth term of an arithmetic sequence	$u_n = u_1 + (n-1)d$

Where u_1 is the first term of the sequence and 'd' is the common difference between each term in the sequence. This is used to find the n^{th} term of a particular arithmetic sequence.

For example, $u_n = 3n - 5$.

Find the following terms of the sequence.

i. u_1 ii. u_{27} iii. u_5

To solve the above, we need to substitute ‘n’ with the term that we wish to find.

Answers:

i. $u_1 = 3(1) - 5 = -2$ ii. $u_{27} = 3(27) - 1 = 80$ iii. $u_5 = 3(5) - 1 = 14$

Exercise A:

Q1. For the following sequences find u_7, u_{100}, u_2.

i. $u_n = 17n - 108$ ii. $u_n = 2(n - 28)$+ 57

Q2. An arithmetic sequence starts 23,36,49,62, ... Find the first term of the sequence to exceed 10,000.

Note:- It is helpful to remember that in a arithmetic sequence $u_{n+1} - u_n = d$. When asked to prove that a sequence is arithmetic, this is one of the methods you are expected to use.

<u>Application of Arithmetic Sequences - Approximating using Arithmetic Sequences</u>

Q. Raina has put an empty egg carton on a weighing scale. Its mass is 32g. When the carton is filled with 12 eggs, the total mass of the eggs and cartons is 743g. Find,

I. The average mass of the eggs that are in the carton
II. Hence, find an arithmetic sequence for u_n, the approximate total mass when ‘n’ eggs have been added to the carton.
III. For what values of n is your model valid?

To solve the above question it is important to go step by step.

I. To find the average mass of the eggs in the carton we need to know the total mass of the eggs and the number of eggs (12) one of which has been given. To find the total mass of the eggs only we subtract 32g (mass of carton) from 743g (mass of carton + eggs). 743 - 32 = 711 (total mass of eggs only). Now we need to divide this by the number of eggs: 711/12 = 59.25g. Each egg weighs approximately 59.25g.
II. To write a general formula for the above word problem we need to understand it in depth. When ‘n’ eggs have been added to the basket, the weight of the eggs would be n(59.25). However, we must not forget to include the weight of the basket as well. Therefore, the formula would be: $u_n = 59.25n + 32$.

III. The value of ‘n’ for which this model would be valid only till we know the capacity of the basket. Since the basket remains constant and we have not been told otherwise, the amount of eggs that can be placed depends on the dimensions of the basket. Therefore, for $0 < n \leq capacity\ of\ the\ basket$, this model is valid.

Exercise:

Q. Parth joins a social media platform. After a week he has 34 friends, and after 9 weeks he has 80 friends on the platform. Find,

I. The average number of friends Parth has made each week from week 1 to 9.
II. Assuming that the total number of friends he has after ‘n’ weeks is an arithmetic sequence, find the model that models the number of Parth’s friends in ‘n’ weeks.

Q. A party event venue advertises all-inclusive venue hire and catering costs of $\$\ 6950$for 50 guests and \$11950 for 100 guests. Assume that the cost of venue hire and catering for ‘n’ guests forms an arithmetic sequence.

I. Write a general formula for u_n.
II. Explain the significance of a) the common difference and b) the constant term
III. Estimate the cost of venue hiring and catering for a party with 95 guests.

Sum of an arithmetic sequence

To find the sum of the n terms in an arithmetic sequence use the formula below.

The sum of n terms of an arithmetic sequence	$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$; $S_n = \frac{n}{2}(u_1 + u_n)$

Q. Determine the sum of the first 8 terms of the arithmetic sequence $u_n = 5n + 27$.

A. In order to do this we simply need to go step by step. First, we shall find u_1. To find this, we substitute 1 in place of ‘n’ in the general formula of the sequence to get 5(1) + 27 = 32. Then we need to find the common difference which can be directly obtained; 27. Now we need to apply the formula and get the sum of the first 8 terms of the sequence. $S_8 = 4\{\ 2(32) + (8-1)27\ \}\ =\ 253(4)\ \ = 1012.$

Geometric Sequences and Series

All sequences cannot have a linear relationship and the ones that do not, are called Geometric Sequences. The formula below helps you determine the n^{th}term of a geometric sequence.

The nth term of a geometric sequence	$u_n = u_1 r^{n-1}$

Where $u_1$is the first term of the sequence and $r$is the common ratio.

For example, $u_n = 5(2)^{n-1}$.

Find the following terms of the sequence.
i. u_2 ii. u_5 iii. u_{14}

To solve the above, we need to substitute 'n' with the term that we wish to find.
Answers:
i. $u_1 = 5(2)^{1-1} = 5$ ii. $u_5 = 5(2)^{5-1} = 5(16) = 80$ iii. $u_{14} = 5(2)^{14-1} = 5(8192)$ =40980

Exercise:

Q1. For the following sequences find u_7, u_{100}, u_{200}.
i. $u_n = 17(5)^{n-1}$ ii. $u_n = 2(3)^{n-1}$

Q2. Show that the sequence is geometric : 5,10,20,40. Find u_n and hence the 15^{th} term.

Q3. A Geometric sequence has $u_4 = -70$ and u_7= 8.75. Find the second term of the sequence.

Q4. A geometric sequence with common ratio 'r' and an arithmetic sequence with common difference 'd' have the same first two terms. The third terms of the geometric and arithmetic sequences are in the ratio 2:1. Find,
I. the possible values of 'r' II. For each value of 'r', find the ratio of the 4^{th} terms

Note:- It is imperative to remember that in a geometric sequence $u_{n+1} \div u_n = r$. If you are asked to prove that a sequence is geometric, one of the methods you are expected to use is this.

<u>Sum of finite Geometric Sequences</u>

The general formula below gives you the sum of the n terms of a finite geometric sequence.

The sum of n terms of a finite geometric sequence	$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \ r \neq 1$

Q. Find the sum of the first 8 terms of the arithmetic sequence $u_n = 5(2)^{n-1}$.

A. In order to do this we simply need to go step by step. First, we shall find u_1. To find this, we substitute 1 in place of 'n' in the general formula of the sequence to get 5(1) = 5. Then we need to find the common ratio which is 2. Now we need to apply the formula and get the sum of the first 8 terms of the sequence. $S_8 = \frac{5(2^8-1)}{2-1} = 640$.

Sum of infinite convergent geometric sequences

The sum of an infinite geometric sequence	$S_\infty = \frac{u_1}{r-1}, \ \lvert r \rvert < 1$

Q. The sum of the first three terms of a convergent infinite geometric series is 19. The sum of the series is 27. Find the first term and the common ratio.

Q. The second term of a convergent infinite geometric series is 8/5. The sum of the series is 10. Show that there are two possible series, and find the first term and common ratio in each case.

Sigma Notation and Arithmetic/ Geometric Series

$$\sum_{start}^{end} = general\ formula$$

The sigma notation is used to find the sum of the series.
Expand and evaluate:

1. $\sum_{k=1}^{5} \quad (11 - 2k)$
2. $\sum_{k=1}^{3} \quad (4k)$

Applications of Geometric Sequences

Geometric sequences have many applications. Growth and Decay, Financial math, Population growth and Annual Depreciation are only a few of them that the IBDP syllabus focuses on.

1. Compound Interest & Annual Depreciation

Compound interest	$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$, where FV is the future value, PV is the present value, n is the number of years, k is the number of compounding periods per year, $r\%$ is the nominal annual rate of interest

Using the formula above we can calculate the amount or the future value of a deposit that is compounded annually. Another method to perform calculations of compound interest is through the Graphic Display Calculator, where you have to plug in values that are given in a word problem and you will receive your answer.

USING TECHNOLOGY FOR FINANCIAL MODELS

- N represents the **number of compounding periods**
- $I\%$ represents the **interest rate per year**
- PV represents the **present value** of the investment
- PMT represents the **payment each time period**
- FV represents the **future value** of the investment
- P/Y is the **number of payments per year**
- C/Y is the **number of compounding periods per year**
- PMT : END BEGIN lets you choose between payments at the end of a time period or payments at the beginning of a time period. Most interest payments are made at the end of the time periods.

Example:

Q. A certain sum amounts to \$ 72900 in 2 years at 8% per annum compound interest, compounded annually. Find the sum.

A.

Let the sum be \$ 100. Then, amount = \$ $\{100 \times (1 + 8/100)^2\}$ = \$ $(100 \times 27/25 \times 27/25)$ = \$ (2916/25)

If the amount is \$ 2916/25 then the sum = \$ 100.

If the amount is \$ 72900 then the sum = \$ $(100 \times 25/2916 \times 72900)$ = \$ 62500.

Hence, the required sum is \$ 62500.

Exercise:

It is known that the number of fish in a certain lake will decrease by 7% each year unless the addition of new fish occurs. At the end of each year, 250 new fish are added to the lake. At the start of 2018, there are 2500 fish in the lake.

A. Show that there will be approximately 2645 fish in the lake at the start of 2020.

B. Find the approximate number of fish in the lake at the start of 2042.

2. Recurring Deposits

One of the applications of Geometric sequences are recurring deposits. As the name suggests in a recurring deposit, a fixed amount of money is invested at a fixed duration for a fixed period of time. These installments all mature on the same date. Essentially, a recurring deposit is like having multiple fixed deposit investments, all of which mature on the same day.

Example:

Q. Parth has started renting an apartment. He paid $5000 rent in the first year, and his rent increased by 5% each year.

a. Find, to the nearest $10, the rent Parth pays in the 5th year
b. Determine an expression for the total rent Parth pays during the first n years.
c. How much rent did Parth pay during his 7 year stay?

A. To solve this we must use our knowledge of sequences and their sums.

a. We need to determine a general formula for the rent she pays. Since the principal amount is 5000 dollar, and the rate is 5% the general formula for the compounded rent is the following:

$$u_n = 5000(1 + \frac{5}{100})^{n-1}$$

To find the amount in the 4th year, simply substitute 5 in place of n.

b. The total rent paid by Paula is essentially the sum of the finite geometric sequence that we derived in part (a). Thus using the formula we can say: $S_n = \frac{5000(1.05^n - 1)}{1.05 - 1}$, which is equivalent to $S_n = 100000(1.05^n - 1)$.

c. To determine how much she paid in the first 7 years we can use the formula that we derived in subpart (b) since it gives us the total amount paid in the first 'n' years. $S_7 = 100000(1.05^7 - 1)$.

Example:

Q. Prashant initially puts $6000 in an account, which earns 5% interest paid annually. At the end of each year, he invests another $1000 in the same account. Find the amount in the account after 8 years.

A. To solve this particular question we need to derive an equation that shows the amount he saves per year. Every year we are told Prashant deposits $6000 and at the end another $1000. If A_n is the amount he has at the end of each year then we can say that :

$$A_1 = 6000(1.05) + 1000$$

The amount he has at the end of the second year would be the interest he earns on the amount he had in the first year, plus the $1000 he deposited.

Thus $A_2 = A_1(1.05) + 1000 = (6000(1.05) + 1000)1.05 + 1000 = 6000(1.05)^2 + 1000(1.05) + 1000$.

We can therefore say that $A_8 = 6000(1.05)^8 + 1000(1.05)^7 + 1000(1.05)^6 + 1000(1.05)^5 + 1000(1.05)^4 + 1000(1.05)^3 + 1000(1.05)^2 + 1000(1.05)^1 + 1000$.

We see that the amount following $6000(1.05)^8$is a geometric series whose common ratio is 1.05. Thus A_8can be written as $A_8 = 6000(1.05)^8 + \frac{1000(1.05^8-1)}{1.05-1}$= \$18413.84 which is the value of the account after 8 years.

Exercise:

Q. On Mary's birthday 1st January 1998, Mary's grandparents put \$*x* in a savings account. They kept on depositing \$*x* on the first day of each month thereafter. The account paid a fixed rate of 0.4% interest every month. The interest was determined on the last day of every month. Then, the interest is added to the account. Let \$$A_n$ be the amount in Mary's account on the last day of the *n*th month, immediately after the addition of the interest.

a) Determine an expression for *A*1 and prove that *A*2 =$1.004^2 x$ + 1.004*x*

b) (i) Find a similar expression for A3 and A4.

(ii) Hence prove that the amount in Mary's account the day before her 10th birthday is given by 251($1.004^{120} - 1$)x.

c) Write down an expression for A_n in terms of x on the day before Mary's 18th birthday, indicating clearly the value of n.

d) Mary's grandparents hoped that the amount in her account would be at least \$20 000 the day before she was 18. Determine the minimum value of the monthly deposit \$*x* required to achieve this and give your answer correct to the nearest dollar.

e) As soon as Mary turned 18, she decided to invest \$15 000 of the money from her grandparents in an account of the same type earning 0.4% interest per month. She withdraws \$1000 every year on her birthday to get herself something. Determine how long she can do so until there is no money left in her account.

Q. Raina takes a bank loan of \$150 000 to buy a house, at an annual interest rate of 3.5%. The interest is calculated at the end of each year and is added to the amount outstanding.

a) Find the amount Raina would owe the bank after 20 years. Give your answer to the nearest dollar.

To pay off her loan, Raina makes annual deposits of \$P at the end of every year in a savings account, paying an annual interest rate of 2% . She makes her first deposit at the end of the first year after taking out the loan.

b) Show that Raina's savings total to $(1.02^{20} - 1)P$ after 20 years .

c) Given that Raina's aim is to own the house after 20 years, find P to the nearest dollar.

Raina visits a different bank and makes a single deposit of \$Q , the annual interest rate being 2.8%.

d) Raina wants to withdraw \$5000 at the end of each year for a period of n years. Prove that an expression for the minimum value of Q is $\frac{5000}{1.028}+\frac{5000}{1.028^2}+\ldots+\frac{5000}{1.028^n}$.

PART C

Laws of Exponents:

1. $a^m \times a^n = a^{m+n} \rightarrow$ Product Law
2. $a^m \div a^n = a^{m-n} \rightarrow$ Division Law
3. $(a^m)^n = a^{mn} \rightarrow$ Power Law
4. $a^0 = 1$
5. $a^{-m} = \frac{1}{a^m} \rightarrow$ Negative Index Law
6. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
7. $(a \times b)^m = a^m \times b^m$
8. $(\frac{a}{b})^n = \frac{a^n}{b^n}$
9. $(\frac{b}{a})^{-n} = (\frac{a}{b})^n$
10. $b = b^c \rightarrow a = c$

PART D

Laws of logarithms:

1. $a = b^c$, then $log_b a = c$
2. $log_a x + log_a y = log_a (xy)$
3. $log_a x - log_a y = log_a (\frac{x}{y})$
4. $log_a x^n = n log_a x$
5. $log_a 1 = 0$
6. $log_a a = 1$
7. $-log_a x = log_a \left(\frac{1}{x}\right)$

8. $log_a b = \frac{1}{log_b a}$

The same rules apply for logarithms with the base e: (ln x).

Change of Base Theorem

$$log_a x = \frac{log_b x}{log_b a}$$

Essentially, it is a formula that permits you to rewrite a logarithm in terms of logs with another base. This is very important when using a calculator to evaluate a log to any base other than 10 or e.

Example:

$$log_{16}32 = \frac{log_2 32}{log_2 16} = \frac{5}{4}$$

Example:

$$log_2 3 = \frac{log_{10}3}{log_{10}2} \simeq \frac{0.47712}{0.30103} \simeq 1.585$$

Example:

$$log_8 x = \frac{ln\ \ x}{ln\ \ 8} = \frac{1}{ln\ 8} ln\ x$$

Note:- Another very important type of question that often comes in the exam is the following.

Q. Solve $4^x + 2^{x+2} = 3$ using logs.

A. To solve this question we must simplify the equation so that the terms with ‘x’ have the same base: $2^{2x} + 2^x(4) - 3 = 0$

B. Next, it is important to realise that this can be written as a quadratic equation where we let $2^x = a$. If we do that we get: $a^2 + 4a - 3 = 0$.

C. Using the quadratic formula we get $a = -2 \pm \sqrt{7}$. However, we do not have to find ‘a’ the questions asks us to find ‘x’, therefore we can rewrite the same as $2^x = -2 \pm \sqrt{7}$.

D. Using logarithms we can rewrite this as $x = log_2(-2 \pm \sqrt{7})$. However, we are not done yet. Since log of a negative number does not exist, $log_2(-2 - \sqrt{7})$ does not exist and should be omitted. The final answer is $x = log_2(-2 + \sqrt{7})$.

Exercise:

1. Solve the simultaneous equations

$$log_2 6x = 1 + 2log_2 y$$
$$1 + log_6 x = log_6(15y - 25).$$

2. Solve $(ln\, x)^2 - (ln\, 2)(ln\, x) < 2(ln\, x)^2$
3. Solve the equation $log_2(x+3) + log_2(x-3) = 4$
4. Find the integer values of m and n for which

$$m - nlog_3 2 = 10log_9 6$$

5. The first terms of an arithmetic sequence are

$$\frac{1}{log_2 x}, \frac{1}{log_8 x}, \frac{1}{log_{32} x}, \frac{1}{log_{128} x}, \ldots$$

Find x if the sum of the first 20 terms is equal to 100.

Using the Graphic Display Calculator one can graph certain equations to solve questions as well.

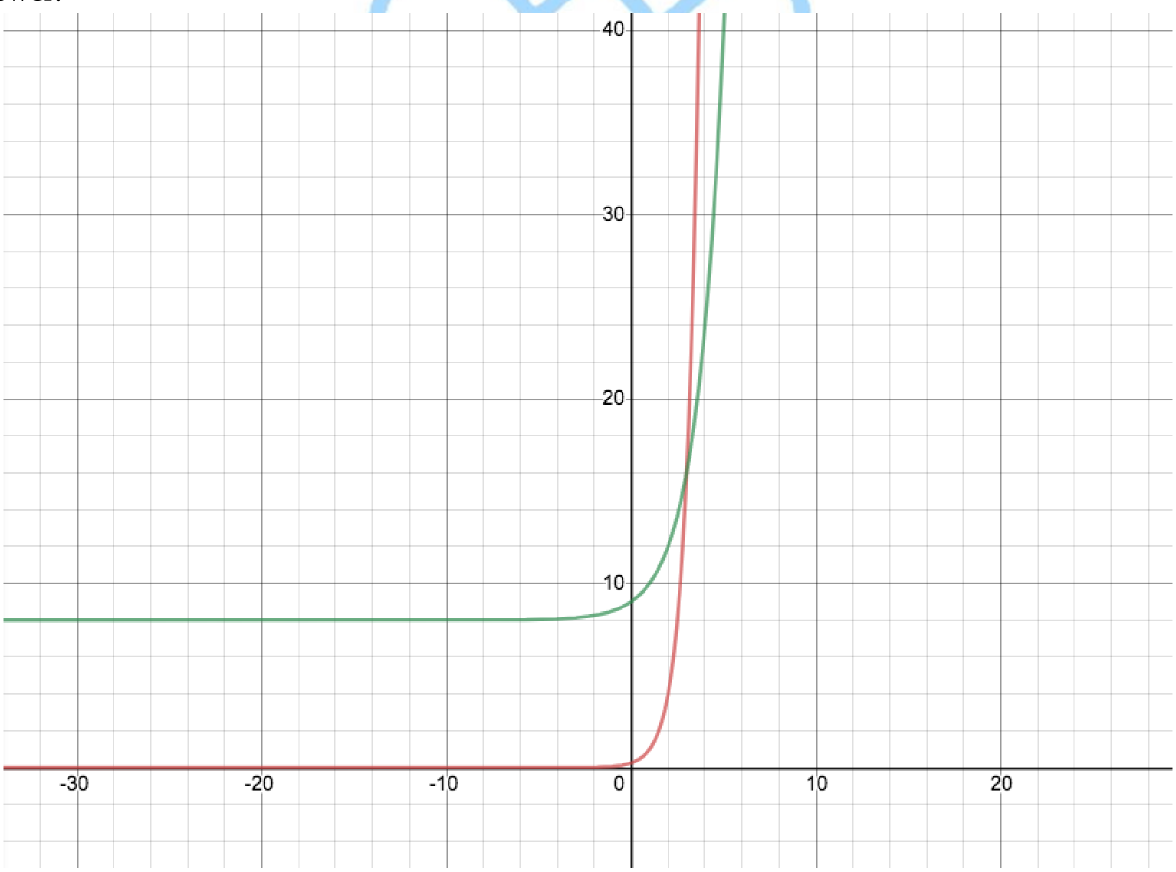
Example 1:

Q. Solve the equation $4^{x-1} - 2^x = 8$

A. To solve this by going to the Graphic function in the GDC we would type in

Y1 = 4^{x-1}

Y2 = 2^x+ 8, and then locate the intersection. The ‘x’ value of the intersection would be the answer.



In this case the intersection point is (3,16). Therefore the answer is x = 3.

PART E

Binomial Expansion:

The binomial theorem uses the concept of Pascal's Triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

And so on and so forth….

Binomial Expansion is the expansion of expressions into the sum of their individual terms.
How to use Pascal's triangle to perform binomial expansion?

When we have to expand -

$$(a+b)^{1} = 1a + 1b$$
$$(a+b)^{2} = 1a^{2} + 2ab + 1b^{2}$$
$$(a+b)^{3} = 1a^{3} + 3a^{2}b + 3ab^{2} + 1b^{3}$$
$$(a+b)^{4} = 1a^{4} + 4a^{3}b + 6a^{2}b^{2} + 4ab^{3} + 1b^{4}$$

..And so on and so forth. We see that there is a pattern, which is the same as in Pascal's triangle and therefore we use Pascal's triangle in Binomial Expansion.
For example, if we want to expand $(x-2y)^{3}$, we can perform binomial expansion and do it. Since the power of the expansion is '3' we use the 4th row of Pascal's triangle to determine the coefficient of each term in the expression. One other important thing to note is that the number of terms 'n' in the expansion is always one more than the power. Similarly, this expansion would have 4 terms.
First we must write down the coefficients of each of the terms.
Note:- If the expansion has a negative sign involved like in the example shown here, the signs of the coefficients will alternate.

$$+1$$
$$-3$$
$$+1$$
$$-3$$

Second, we write ‘x’ and ‘2y’ next to each of these coefficients. The powers for ‘x’ keep reducing from 3 to 0 and for 2y the powers keep increasing from 0 to 3. Then we simplify each of the terms.

$$+1 \times (x)^3 \times (2y)^0 = x^3$$
$$-3 \times (x)^2 \times (2y)^1 = -6x^2y$$
$$+1 \times (x)^1 \times (2y)^2 = 4xy^2$$
$$-3 \times (x)^0 \times (2y)^3 = -24y^3$$

Thus, we can say that $(x - 2y)^3 = x^3 - 6x^2y + 4xy^2 - 24y^3$.

To determine the value of the coefficients according to Pascal’s triangle we use the concept of Combinations.

$$^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

For example, if we want to know the 3rd term in the Pascal’s Triangle for the 4th power. By looking at the triangle above we know it would be 4, however using this notation we can calculate it.

$$^{4}C_3 = \frac{4!}{3!\,(4-3)!} = 4$$

Similarly we can write $(a+b)^n = \ ^{n}C_0 a^n b^0 + \ ^{n}C_1 a^{n-1} b^1 +$
$^{n}C_2 a^{n-2} b^2 \ldots + \ ^{n}C_n a^0 b^n$

The above is equivalent to the formula given in the Math AA HL formula booklet :

$$(a+b)^n = a^n + {}^{n}C_1 \, a^{n-1}b + \ldots + {}^{n}C_r \, a^{n-r}b^r + \ldots + b^n$$

<u>Note:</u>- It is helpful to remember that $^{n}C_0 = 1$, $^{n}C_n = 1$, $^{n}C_1 = n$, $^{n}C_2 = \frac{n^2-n}{2}$.

To find a specific term in the expansion of an expression, rather than expanding the entire expression we can use a general formula that will help us find the term.

General Formula : $T_{r+1} = \ ^{n}C_r \ a^{n-r}b^r$. Below are a few examples of what kind of questions appear in the exam.

Example:

Write down and simplify the expansion of $(2+x)^4$ in ascending powers of x. Hence find the exact value of $(2.1)^4$.

A. To solve this we first follow the steps shown above.

$$+1 \times (2)^4 \times (x)^0 = 16$$

$$+4 \times (2)^3 \times (x)^1 = 32x$$
$$+6 \times (2)^2 \times (x)^2 = 24x^2$$
$$+4 \times (2)^1 \times (x)^3 = 8x^3$$
$$+1 \times (2)^4 \times (x)^4 = 16x^4$$

Thus, $(2 + x)^4 = 16 + 32x + 24x^2 + 8x^3 + 16x^4$. In order to find the exact value of $(2.1)^4$we must write it in a manner that is easier to compute.

$$(2.1)^4 = (2 + 0.1)^4 Using\ the\ expansion\ obtained$$
$$: 16 + 32(0.1) + 24(0.1)^2 + 8(0.1)^3 + 16(0.1)^4 = 19.4481$$

Example:

Find the term in x^5in the expansion of $(3x + A)(2x + B)^6$.

A. To solve this we need to first expand the expression $(2x + B)^6$ and then multiply it with $(3x + A)$.

$$(2x + B)^6 \quad =$$
$$+1 \times (2x)^6 \times (B)^0 = 64x^6$$
$$+6 \times (2x)^5 \times (B) = 192x^5B$$
$$+15 \times (2x)^4 \times (B)^2 = 240x^4B^2$$
$$+20 \times (2x)^3 \times (B)^3 = 160x^3B^3$$
$$+15 \times (2x)^2 \times (B)^4 = 60x^2B^4$$
$$+6 \times (2x)^1 \times (B)^5 = 12xB^5$$
$$+1 \times (2x)^0 \times (B)^6 = B^6$$

$$(3x + A)(2x + B)^6$$
$$= (3x + A)(64x^6 + 192x^5B + 240x^4B^2 + 160x^3B^3 + 60x^2B^4 + 12xB^5 + B^6)$$

To find the coefficient of x^5we need to multiply only those terms in the two brackets that would give x^5. So, $(3x)(240x^4B^2) + (A)192x^5B = x^5(720B^2 + 192AB)$and therefore the coefficient of $x^5$is $720B^2 + 192AB$.

Example:

Find the constant term in the expansion of $(x - \frac{2}{x})^4$.

A. To find the constant term in the expansion of the above expression we can use the general formula.

$T_{r+1} = \ {}^4C_r \ (x)^{4-r}(2x^{-1})^r$; To find the constant term we need the power of ‘x’ to equate to 0. Since $x^0 = 1$. Through this we can determine the value of ‘r’ and substitute it in the general formula to get the constant term.

Equation the power of ‘x’ to 0:

$$(x)^{4-r}(x^{-r}) \quad = x^0$$
$$4 - r - r = 0$$
$$4 - 2r = 0$$

$$4 = 2r\ ;\ r = 2$$

Now, we can substitute this in the general form equation to get:

$T_{2+1} = {}^{4}C_{2}\ (x)^{4-2}(2x^{-1})^{2} = 6(x)^{2}(\frac{4}{x^{2}}) = 6(4) = 24$. Thus, the constant term in the expansion of $(x - \frac{2}{x})^{4}$is 24.

Exercise:

Q. Expand the following expressions using binomial expansion:-

1. $(2x + 5)^{5}$
2. $(3x + 4)^{7}$
3. $(2x^{2} + 5)^{3}$
4. $(x + 9)^{8}$
5. $(7x + 2)^{14}$

Q. Find the coefficient of x^{8}in the expansion of $(x^{2} - \frac{2}{x})^{7}$.

Q. Find the constant term in the expansion of $(4x^{2} - \frac{3}{2x})^{12}$.

Q. Consider the expansion of $(1 \quad + \quad x)^{n}$where $n \geq 3$.

a. Write down the first four terms of the expansion.

The coefficients of the second, third and fourth terms of the expansion are consecutive terms of an arithmetic sequence.

b. Show that $n^{3} - 9n^{2} + 14n = 0$.
c. Hence find the value of n.

Q. When $(1 + \frac{x}{2})^{n}, n \in N$, is expanded in ascending powers of x, the coefficient of $x^{3}$is 70.

a. Find the value of n.
b. Hence, find the coefficient of x^{2}.

PART F

Partial Fractions:

Writing a fraction in the form of partial fractions makes it easier to integrate. For now, we will understand how to write fractions in the form of partial fractions.

Example:

Q. Write $\frac{2x \quad -8}{x^{2}-4}$ as partial fractions.

A. To do this we first find simplify the denominator.

$$x^{2} - 4 = (x - 2)(x + 2)$$

We can then write it in the form shown below-

$$\frac{2x \quad -8}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$\frac{2x \quad -8}{(x-2)(x+2)} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$; We can now equate $2x - 8 = A(x + 2) + B(x - 2)$.

$$2x - 8 = Ax + 2A + Bx - 2B$$

$2(x) - 8 = x(A + B) + 2A - 2B$; Thus, we can say that $A + B = 2$; $2A - 2B = -8$.

Using the simultaneous equations we get A = -1 and B = 3. Thus we can write $\frac{2x \quad -8}{x^2-4} = \frac{3}{x+2} - \frac{1}{x-2}$.

Exercise:

Q. Write $\frac{20}{(2x-3)(x+1)}$ as the sum of partial fractions.

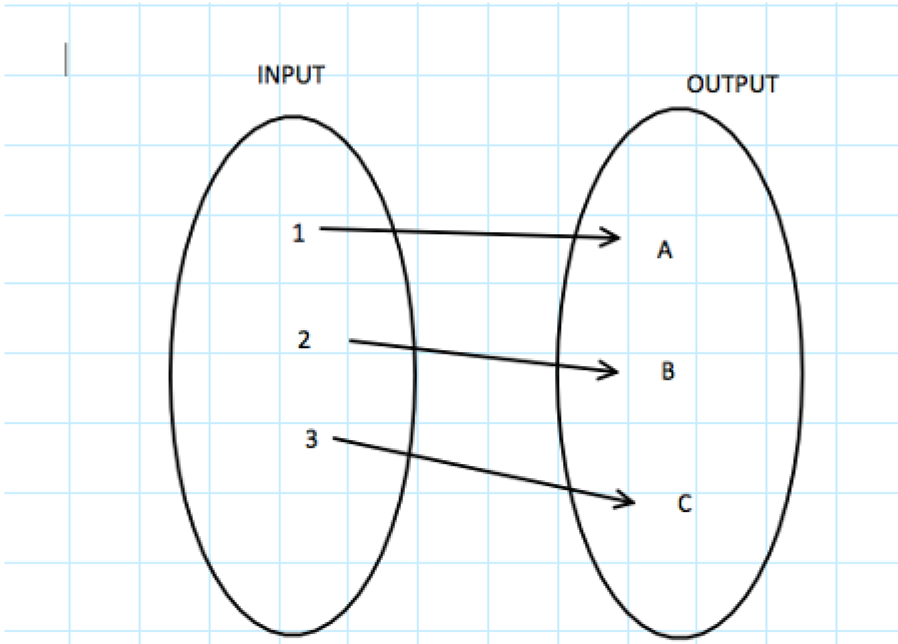
Q. Write $\frac{x-9}{x^2-2x-3}$ as the sum of partial fractions.

Q. Write $\frac{6x^2+x-19}{(x+3)(x-1)^2}$ as the sum of partial fractions in the form $\frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{x-2}$

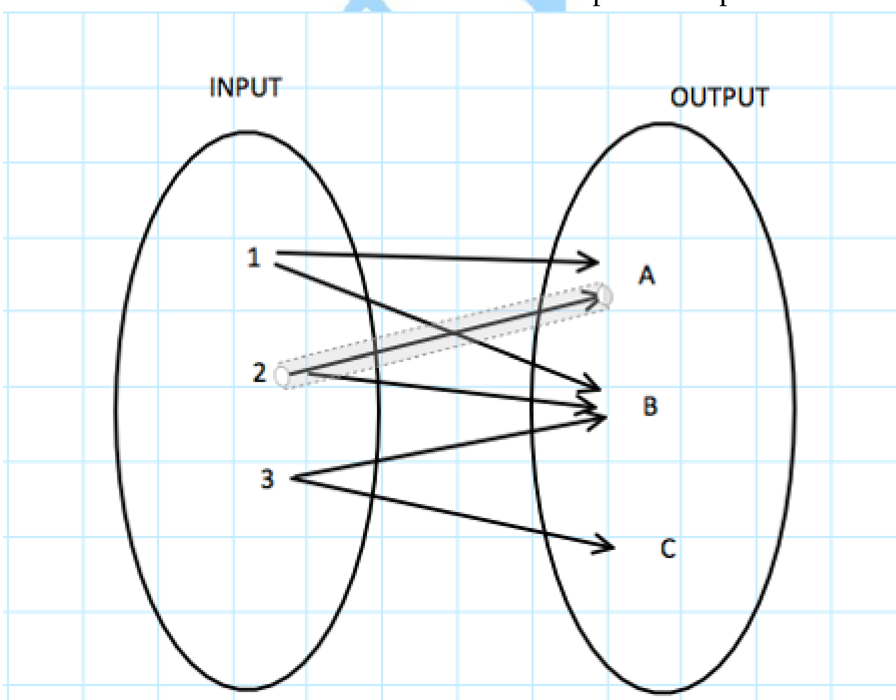
Topic 2: Functions

PART A

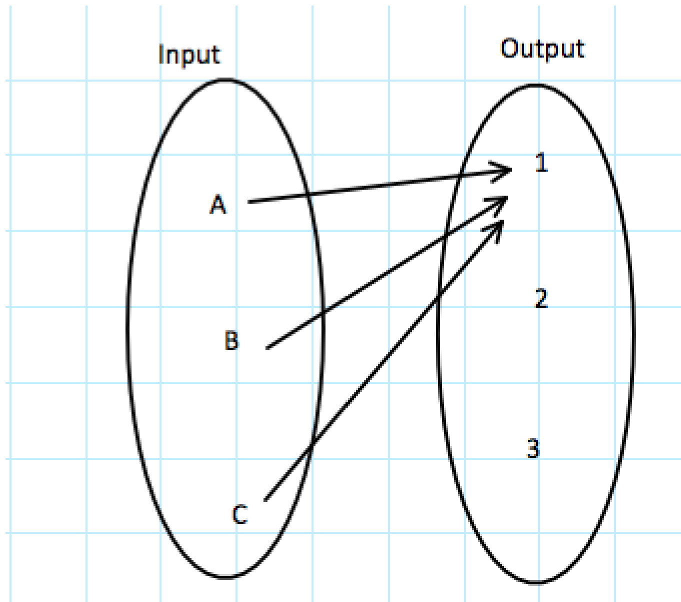
Types of Relations:



The above relation is a one-to-one relation as one input corresponds to one output.



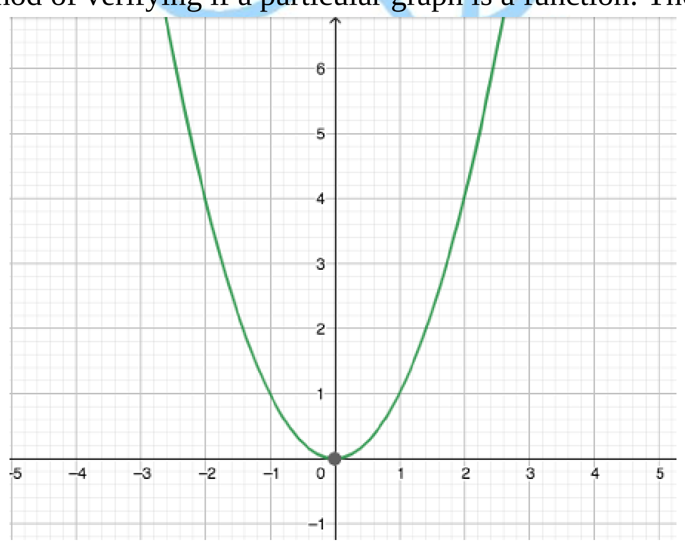
This is a one-to-many relation where one input corresponds to one or more outputs.



This is a many-to-one relation where many inputs correspond to the same specific output.
There are also many-to-many relations, but they are not relevant to our subject.

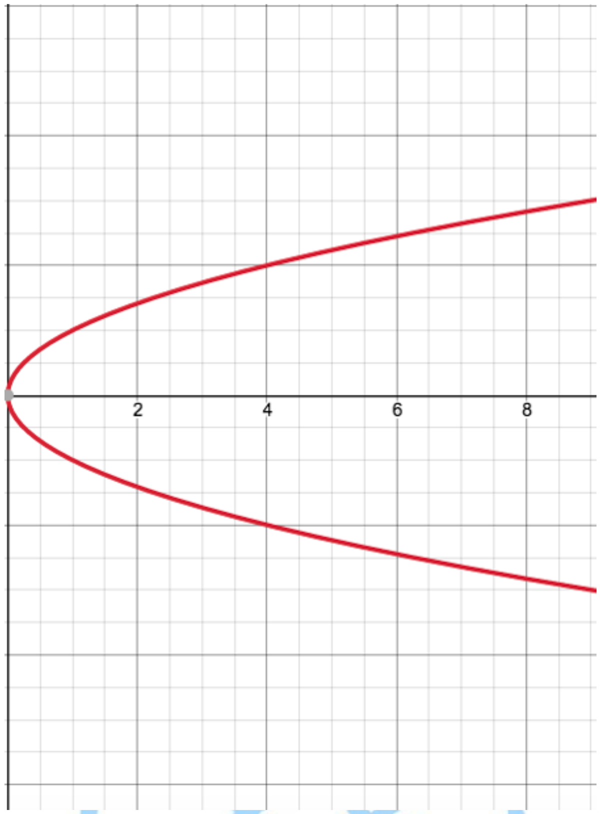
What is a function?

A function is a mathematical method of relating one or more inputs to a specific output.
For example: (Taking x as the input and y as the output)
$y = x^2$ is a function as one or two x-values correspond directly to one specific y-value, however, $y^2 = x$ is not a function as one x-value can correspond to one or more y-values. Essentially, only one-to-one relations and many-to-one relations are functions.
There is a simple method of verifying if a particular graph is a function: The Vertical Line test.



The graph above is given by $y = x^2$ and if we draw a vertical line through the graph at any x-value, we will find that it corresponds to one specific y-value. For example, if we draw a line at x= -2, we get a y-value of 4 and if we draw a line at x=1, we get a y-value of 1. Thus, there is a specific output for every input.

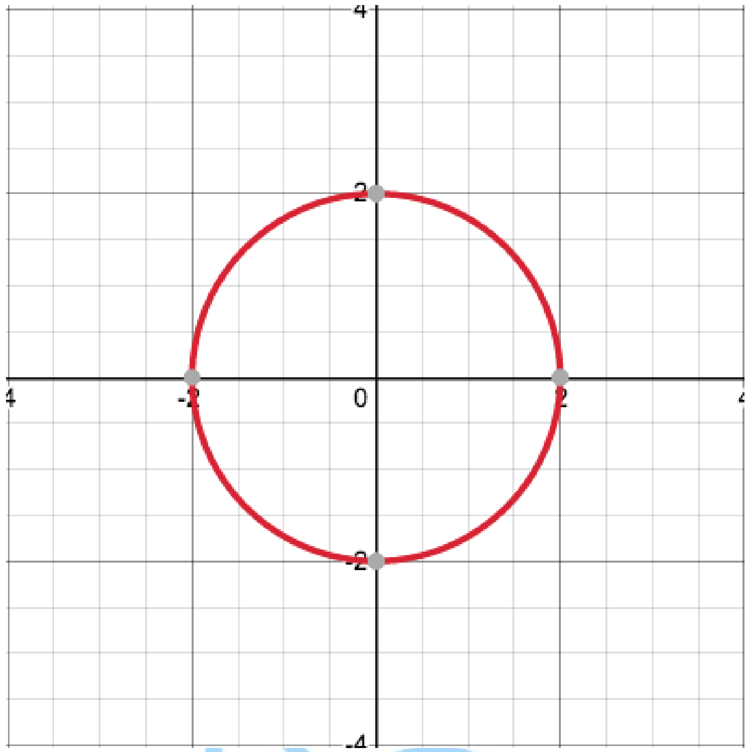
Now, let us look at the graph of $y^2 = x$

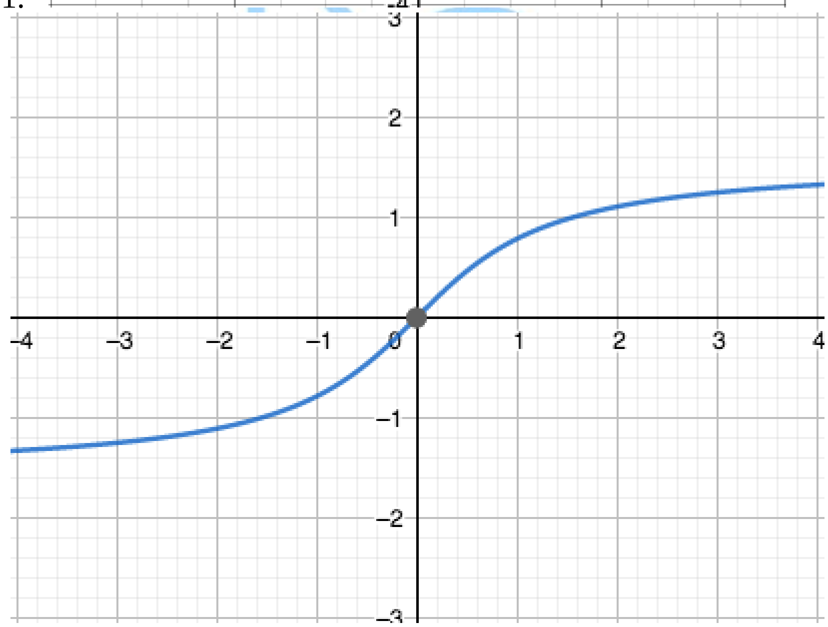


Here, if we draw a vertical line at any x-value in the graph above, it will correspond to 2 outputs. The graph above illustrates a one-to-many relation. This is not a function as functions can only exist as one-to-one and many-to-one relations.

Exercise:

Is this graph a function?

1. 

2. 

PART B

Straight-Line Graphs:

There are 3 different forms of straight-line graphs.

1. $y = mx + c$ (gradient-intercept form).
2. $ax + by + d = 0$ (general form).
3. $y - y_1 = m(x - x_1)$ (point-gradient form).

For two parallel lines:
Gradient of Line 1= Gradient of Line 2
This can be simplified to be written as $m_1=m_2$

For two perpendicular lines:

$$Gradient\ of\ Line\ 1 \times Gradient\ of\ Line\ 2 = -1$$

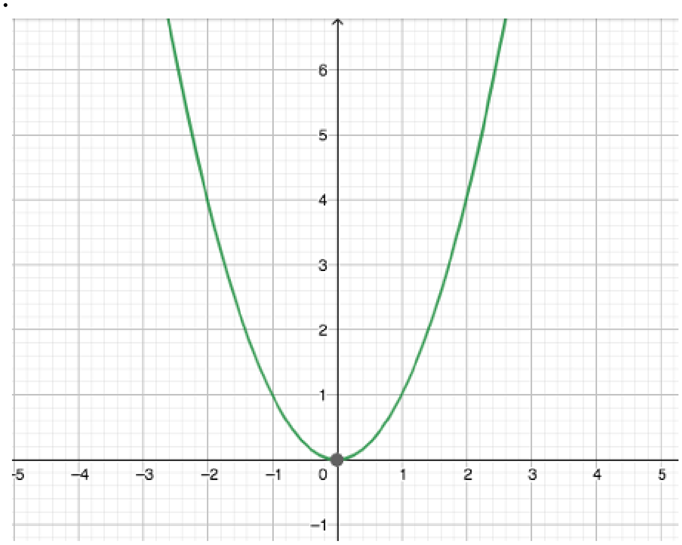
PART C

Domain and Range of Functions and Inverse Functions:

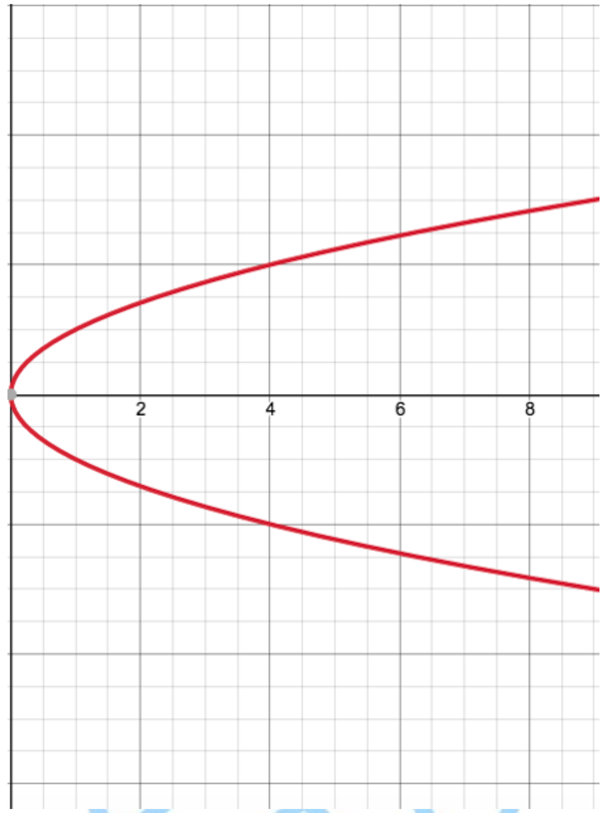
The domain of a function consists of all the possible inputs/x-values that can make a function "work" and will result in the output of real y-values. For example, the function $y = \sqrt{3-x}$ has the domain $x \leq 3$ as the square root of a negative number is not real.
The range of a function consists of all the possible output values that result from the substitution of the domain into the function. For example, for the same function $y = \sqrt{3-x}$, the y-value/output is always greater than zero. Thus, the range can be written as $y \geq 0$.
The inverse of a function is a function that undoes the action of another function. Graphically, this means that the inverse function is a reflection of the original function in the line y=x. In order to find the inverse of a function, we must swap y and x and then try to obtain y in terms of x again.
For example, finding the inverse of $y = x + 3$.
Swapping x and y:

$$x = y + 3$$
$$y = x - 3$$

Thus, $y = x - 3$ is the inverse of $y = x + 3$. This same principle can be applied to all functions. However, for the inverse of a function to be a function, the graph of the original function must pass the Horizontal Line Test
Let us consider $y = x^2$.



If we draw a horizontal line through any y-value of this graph, it will pass through 2 points. This is a problem as we eventually swap x and y while calculating the inverse of this function and that would give us a graph like the one shown below.

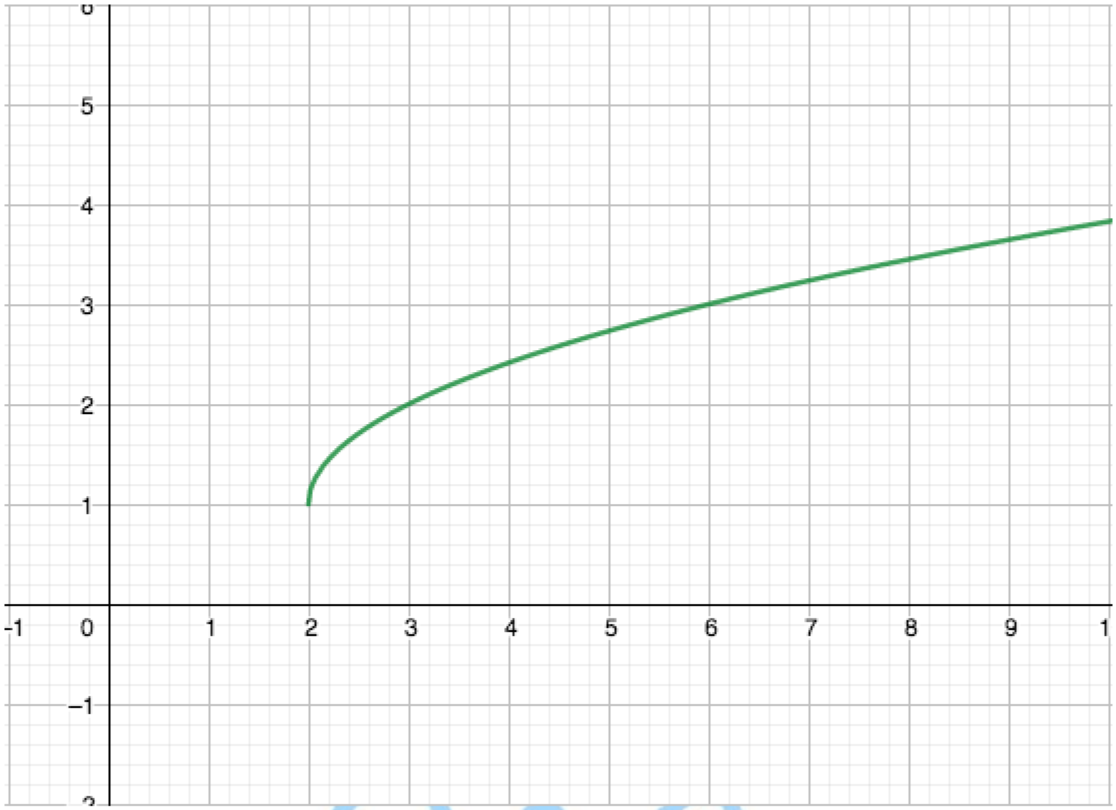


As discussed earlier, this graph is not a function and thus, the inverse of $y = x^2$ for all x-values is not a function.

However, the inverse of $y = x^2$ for $x \geq 0$ is a function as all horizontal lines passing through the y-axis only pass through one output value. Thus, by restricting the domain of a function, we can ensure that the inverse of the function is a function as well.

Exercise:

1. What is the domain and range of the function given below?



2. Find the inverse of this function and state and explain whether the inverse is a function or not.

$$y = (x - 1)^4 - 4$$

PART D

Quadratic Functions:

A quadratic function can be expressed in 3 forms.

1. The standard form: $y = ax^2 + bx + c$
2. The vertex form: $y = a(x-h)^2+k$ where (h, k) is the vertex
3. The intercept form: y = a.(x-p).(x-q) where x=p and x=q where y=0, that is, p and q are the x-intercepts.

<u>**Note-**</u> 'a' is the vertical stretch factor.

<u>Converting from one form to another:</u>

From standard form to vertex form:

There are 2 possible ways of converting equations from standard form to vertex form. The first uses a method known as completing the square. Let us consider the equation $y = x^2 - 2x + 2$. In this expression, a = 1. Thus, the method can be applied immediately.

Steps:

1. Take the coefficient of the middle term, in this case, 2, divide it by 2 and square it. For this case, that gives us 1.
2. The answer from step 1 is added and subtracted on the same side of the equation. This is done because simply adding 1 on the same side of the expression will change its meaning, thus, to keep it the same, the same value must be subtracted on the same side
3. This gives us $x^2 - 2x + 2 + 1 - 1$. Let us group together the perfect square expression, which is $(x^2 - 2x + 1)$ and simplify the other two terms. Hence, we get $y = (x^2 - 2x + 1) + 1$.
4. With further simplification, we obtain the equation in the vertex form: $y = (x-1)^2+1$.

The second method is much faster and can be used when you are more comfortable with quadratic equations. Let us consider the same equation $y = x^2 - 2x + 2$. Here, a =1, b=-2 and c=2. To find the x-coordinate of the vertex, we can use the relation $x = \frac{-b}{2a}$.Then, we can obtain the y-value of the vertex by plugging in the x-value obtained from the above relation into the equation. Finally, we write the equation in the vertex form $y = a(x-h)^2+k$ using the values of 'h' and 'k' we just obtained.

From vertex form to standard form:

Simple expansion of the equation is required in order to convert the vertex form to the standard form. Considering the example of $y= (x-3)^2+2$, we first square the term within the brackets. This gives us $x^2 - 6x + 9 + 2$ and that simplifies to $x^2 - 6x + 11$.

From standard form to intercept form:

$y = x^2 - 3x + 2$. The middle term is the sum of the two zeroes and the y-intercept 'c' is the product of the two zeroes. Thus, the two x-intercepts are x=1 and x=2 as they satisfy the two conditions mentioned earlier.

From intercept form to standard form:

Simple expansion of the equation is required in order to convert the intercept form to the standard form. Considering the example of $y= 2(x-1)(x-2)$, we first open multiply the terms within the brackets and then multiply all the terms in the simplified expression by 2. That will give a final equation as $y= 2x^2 - 6x + 4$.

From intercept form to vertex form:

Let us take the equation: $y = (2x - 1)(x + 2)$. We will simplify it to :$y = 2(x - 1/2)(x + 2)$. This means that x_1 (the first x-intercept) =½ and x_2 (the second x-intercept)= -2. Now, we shall add up x_1 and x_2. This gives us -3/2. This value is also equal to -b/a. Thus, to obtain the x-value for the vertex, we simply divide -3/2 by 2 and replace x by -¾ in our initial equation to get the y-value of the vertex. Thus, we can rewrite the equation in the vertex form, using our calculations.

From vertex form to intercept form:

There are two possible methods. We can reverse the methodology mentioned above and work backwards to obtain our equation in the intercept form. Alternatively, you can simply expand the vertex form to obtain the standard form and then use the standard form to intercept form conversion as mentioned earlier!
Note: Sometimes, certain equations do not have x-intercepts and we will deal with them in depth later on.

Domain and Range of Quadratic Equations:

If we do not want to find the inverse of a quadratic function, its domain consists of the set of all real numbers. Contrarily, if we want to find the inverse function of a quadratic function, we must consider the x-coordinate of the vertex. If the inverse function is to be negative, then $x \leq h$ and if the inverse function is to be positive, then $x \geq h$. The range of quadratic equations, however, depends on the sign of the 'à' value and the y-coordinate of the vertex. If $a > 0$, the graph has a minimum value and thus, the range is $y \geq k$. However, if $a < 0$, the graph has a maximum value and thus, the range is $y \leq k$.

Using the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The above formula can be used with the standard form where y = ax^2 + bx+ c.
The part of this formula that is of particular interest to us is $b^2 - 4ac$

Let us understand the importance of this part, formally known as the discriminant.
If $b^2 - 4ac$ >0, then a quadratic equation has 2 real, different roots (x-intercepts)
If $b^2 - 4ac$ =0, then a quadratic equation has 2 real, equal roots (x-intercepts)
If $b^2 - 4ac$<0, then a quadratic equation has no real roots and thus, has no x-intercepts.

A very common question where utilising the discriminant is vital:

$$3kx^2 + 2x + k = 0. \textit{Find the possible values of } k \textit{ that will give 2 distinct, real roots or 2 equal, real roots or none,}$$

Here, we utilise the discriminant: $b^2 - 4ac$. We take '2' as our 'b' value, 'k' as our c value and '3k' as our 'à' value and use them to solve for the possible values of 'k' depending on the presence of real roots.

Concavity:

Most functions have a type of concavity. The concavity of quadratic functions depends on the sign of "a" in y = ax^2 + bx+ c. If "a" is positive, then the function is positive and the graph of 'f' is concave up. If "a" is negative, then the function is negative and the graph of 'f' is concave down.

Exercise:

Convert the following equations from the standard form to the vertex form, find its domain and range and simultaneously determine if the function has real roots:

1) $y = x^2 + 6x + 5$
2) $y = x^2 + 3x + 2$
3) $y = 3x^2 + 12x + 19$

Convert the following equations from the vertex form to the standard form, find its domain and range and simultaneously determine if the function has real roots:

1. $y = 3(x + 5)^2 - 59/5$
2. $y = 1/5\ (x - 1)^2 + 1$
3. $y = 5/2\ (x - 1/2)^2 + 6$

Convert the following equations from the standard form to the intercept form, find its domain and range and simultaneously determine if the function has real roots:

1. $y = x^2 + 13x + 36$
2. $y = 2x^2 + 8x + 6$
3. $y = 4x^2 - 9x + 2$

Convert the following equations from the intercept form to the standard form, find its domain and range and simultaneously determine if the function has real roots:

1. $y = 2(x + 1)$(x-5)
2. $y = (x + 1/9)\ (x - 3)$
3. $y = 3(x + 2)(1 - x)$

Convert the following equations from the intercept form to the vertex form, find its domain and range and simultaneously determine if the function has real roots:

1. $y = 2(x + 1/2)\ (x - 3)$
2. $y = (x - 1)\ (x + 3)$
3. $y = 8(x + 1)(x - 2)$

Convert the following equations from the vertex form to the intercept form, find its domain and range and simultaneously determine if the function has real roots:

1. $y = (x + 2)^2 - 1$
2. $y = (x - 3)^2 - 1$
3. $y = 2(x - 1.25)^2 - 6.125$

PART E

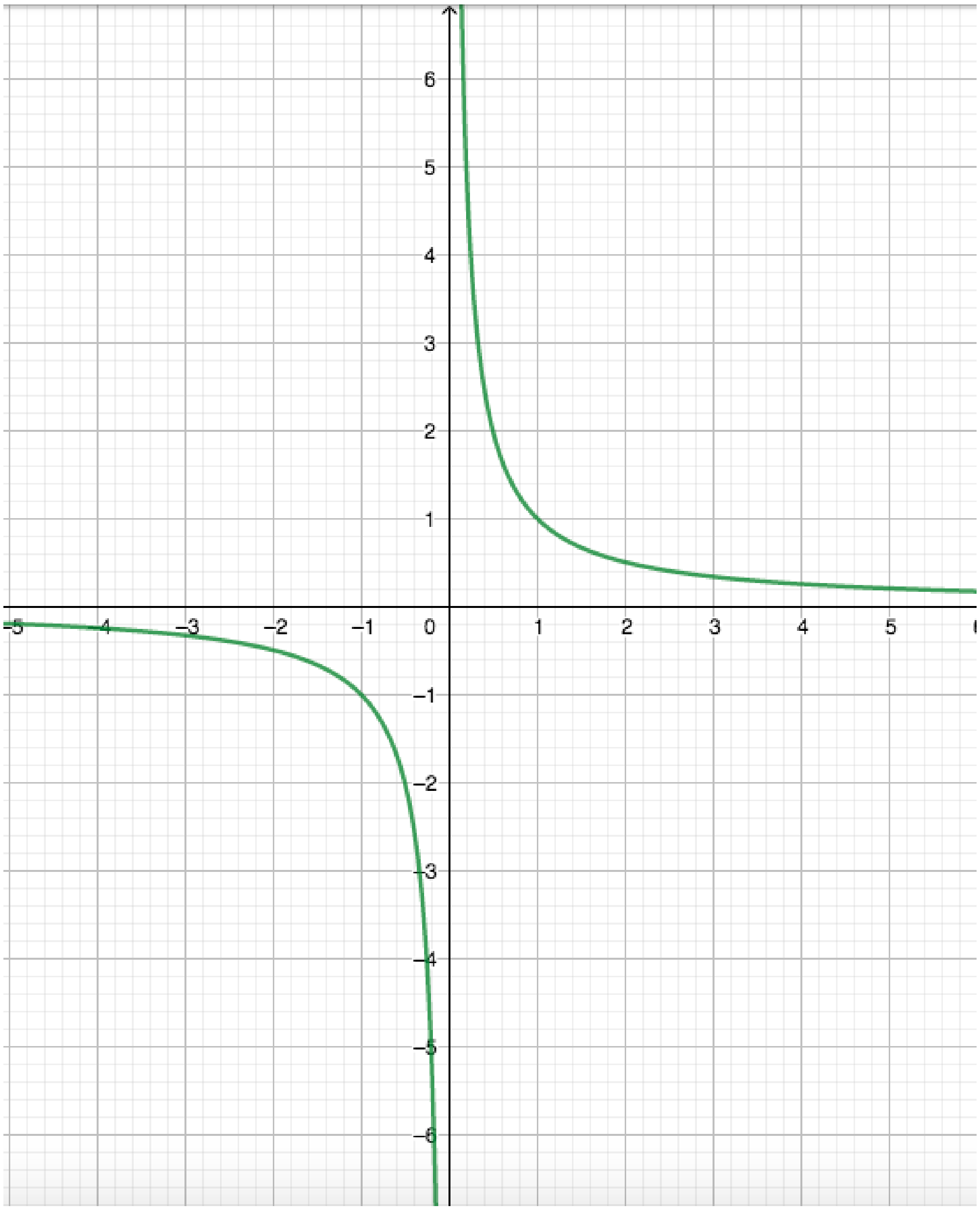
Graphing Functions and Using Graphs to obtain key data:

Drawing a function:

Let us consider the function $y=x^2$. In order to draw the graph of $y=x^2$, we must actually calculate the y-value of the function for every x-value, plot the points on a graph paper and draw the graph.

Sketching a function:

This is only an approximation of what the graph of a function will actually look like. For sketching a function, some key features that must be considered are the x-intercepts (roots), y-intercepts, which can easily be calculated. While graphing functions, another key feature that must be considered is an asymptote. An asymptote is a value that the graph of a function keeps on getting closer and closer to, but never quite reaches. The most common example of asymptotes can be found in functions where 'x' is in the denominator. For example, let us consider $y = \frac{1}{x}$. At x=0, the value of y is infinite, which is not possible. Thus, the y-value approaches negative infinity as the x-value approaches zero from the left hand side and approaches positive infinity as the x-value approaches zero from the right hand side. This is perfectly illustrated in the graph below.



Other important aspects of a graph that must be considered are the maximum and minimum values. While some functions do not have maximum and minimum values, most do and they must be clearly illustrated in a graph. We have previously discussed maximum and minimum values through the vertex of a quadratic function. However, it is worth noting that the maximum and minimum values do not always have to determine the range of the function. Additionally, another key aspect we can find out from the graphs of functions, is their symmetry. Referring back to quadratic functions, their axis of symmetry lies along the x-value of their vertex as if the graph of the function is folded along that x-value, it will completely overlap.

Exercise:

Sketch the following graphs, keeping in mind the features previously discussed:

1. y=(x+3)(x-2)(x+5)
2. y= -(2x+1)^2
3. $y = \frac{1}{x-2}$

PART F

Composite Functions:

A composite function treats the output of one function as the input for another. This is represented by the notation $(fog)(x)$ or even f(g(x)). Let us consider an example.

$$g(x) = x^3$$
$$f(x) = 2x + 3$$
$$(fog)(x) = 2x^3 + 3$$
$$(gof)(x) = (2x + 3)^3$$

This clearly illustrates that $(fog)(x)$is not necessarily equal to $(gof)(x)$.
Additionally, composite functions only exist if the range of the first function is a subset of the domain of the second function.
Another key aspect to note is that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. This is a direct link to the content covered in Topic 2, Part B.

Exercise:

Find the composite functions, using f(x) and g(x) as noted below, and check if the composite function exists.

$$f(x) = \sqrt{x + 3}$$

$g(x) = 5x^2 + 6$

1. $(fof)(x)$
2. $(gog)(x)$
3. $(fog)(x)$
4. $(gof)(x)$

PART G

Rational Functions:

A rational function is any function that can be expressed as a quotient $y = \frac{f(x)}{g(x)}$, g(x) is not equal to zero, where f(x) and g(x) are polynomial functions.
Steps to graph rational functions:

1. Find the x-intercept by equating f(x) to zero
2. Find the y-intercept by substituting x=0 in both f(x) and g(x)
3. Find the vertical asymptote by equating g(x) to zero
4. Find the horizontal asymptote by dividing the coefficient of x in the numerator by the coefficient of x in the denominator

Some rational functions have oblique asymptotes. We can tell if a function has an oblique asymptote if we compare the highest power in the numerator to the highest power in the denominator. If the highest power in the numerator>highest power in the denominator, then the function has an oblique asymptote. The Math AA syllabus only includes functions of the type $f(x) = \frac{ax^2 + bx + c}{dx + e}$ .To find the oblique asymptote, we divide the numerator by the denominator using long division.The quotient is the oblique asymptote. Additionally, this type of rational functions do not have horizontal asymptotes.

However, rational functions of the type $f(x) = \frac{ax + b}{cx^2 + dx + e}$, have their horizontal asymptote at y=0. This can be generalised for all functions where the degree of the numerator of f(x) is less than the degree of the denominator. However, if there is a vertical transformation 'b'(mentioned in Part J), then the horizontal asymptote is y=b.

Exercise 2G:

Sketch the rational functions listed below, using the steps outlined above.

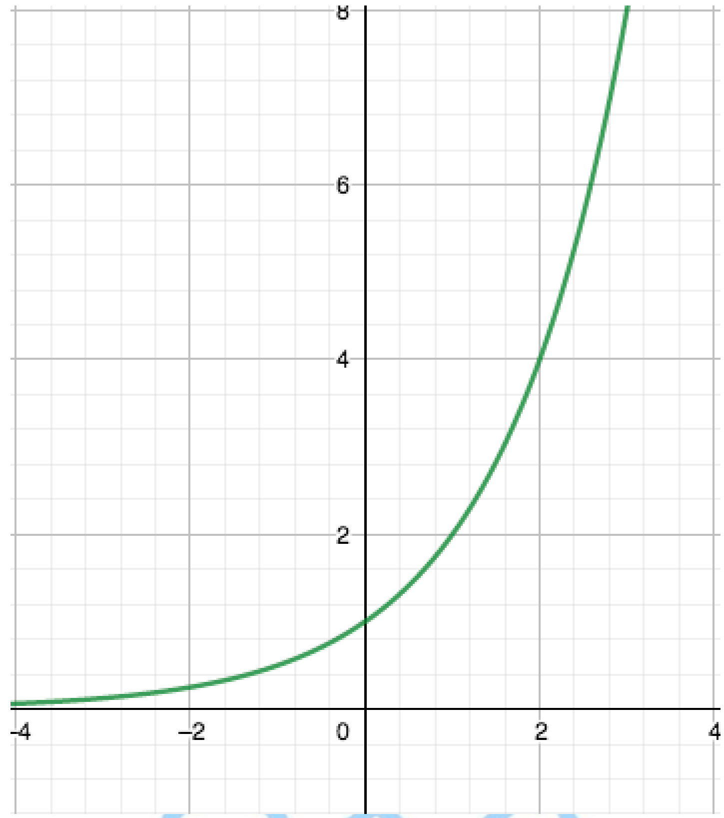
1. $y = \frac{1}{2x-3}$
2. $y = \frac{x+4}{x-2}$
3. $y = \frac{2x+3}{x-1}$

PART H

Exponential and Logarithmic Functions:

Exponential Functions are expressed in the form: f(x) = a^x , a > 0, or e^x.

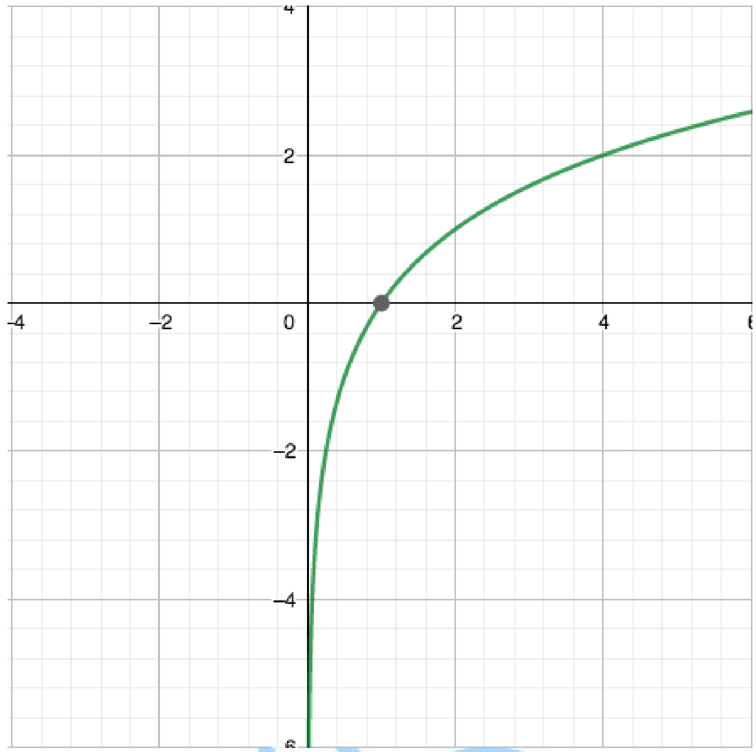
Let us graph 2^x. There is no x-intercept and there is a y-intercept at (0,1). The y-value doubles every time the x-value increases by 1. The graph obtained is given below:



All exponential functions have a y-intercept at (0,1) and will approach 0 as the x-value gets infinitely more negative as seen above. There is an exception to this rule, which will be mentioned later on.

Logarithmic Functions are expressed in the form $f(x) = \log_a x$, $x > 0$ or $f(x) = \ln x$, $x > 0$.

Let us graph $\log_2 x$. There is a x-intercept at (1,0) and there is no y-intercept. Every time the x-value doubles, the y-value increases by 1.(exactly the opposite of 2^x, or mathematically, its inverse). The graph obtained is given below.



All logarithmic functions have a x-intercept at (1,0) and will approach negative infinity as the x-value approaches 0.

Exercise 2H:
Graph the following exponential and logarithmic functions:

1. $y = ln(x)$
2. $y = 4^x$
3. $y = log(x)$
4. $y = 1^x$

PART I

Solving equations:

Let us try and solve the equation given below.

$$2^{2x} - 4.2^x + 4 = 0$$

There are 2 methods that can be used. Firstly, we can simply graph this function on a calculator or a graphing software like Desmos and obtain the answer by checking the roots of the function. Alternatively, we can equate 2^x to b and rewrite the equation as a quadratic equation, which is noted below:

$$b^2 - 4b + 4 = 0$$

Once we have this equation, we can easily solve it to get the answer. Then, we can resubtitute that value of b in its relation to 2^x and get the value of x.

However, there are some equations that we cannot solve algebraically and thus, must use technology to find their solution. One such equation is $e^x = cos(x)$. Here, we simply graph e^x and cos(x) and find their intersection on a calculator or graphing software.

Exercise:

Solve the following equations, either algebraically or graphically:

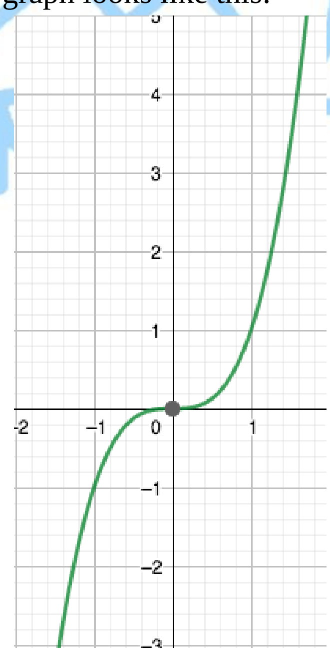
1. $e^{-x} + e^x = 4$
2. $ln(x) = tan(x)$

PART J

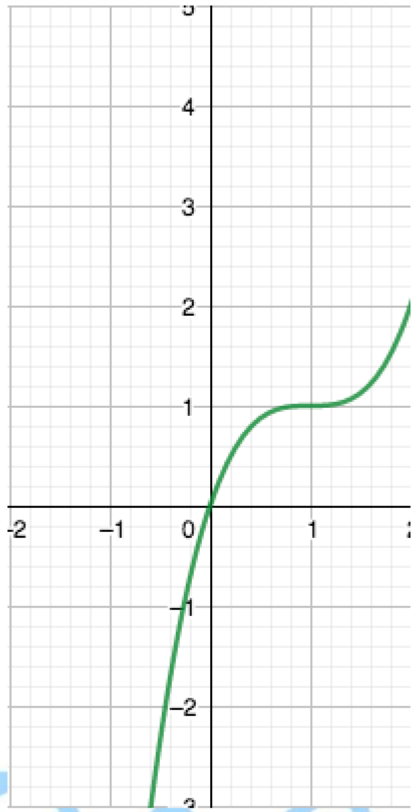
Basic Transformations of Functions:

Transformations of functions are used to shift and/or resize the graphs of functions. There are 3 types of transformations that we will be looking at in depth.

1. Translations: $y = f(x) + b$; $y = f(x - a)$. The first translation mentioned, f(x)+b essentially moves the graph of a function up or down along the y−axis. For example, if $f(x)=x^2$ and b = -1, then the function x^2 is moved down and has a y-intercept of -1 and there is a change in the x-intercepts as well. The second translation has the same principle, but it moves the graph of a function left or right along the x-axis. So if a>0, the graph is translated to the right by 'à' units and if a<0, the graph is translated to the left by 'à' units. Let us consider an example. $f(x) = x^3$.The initial graph looks like this:

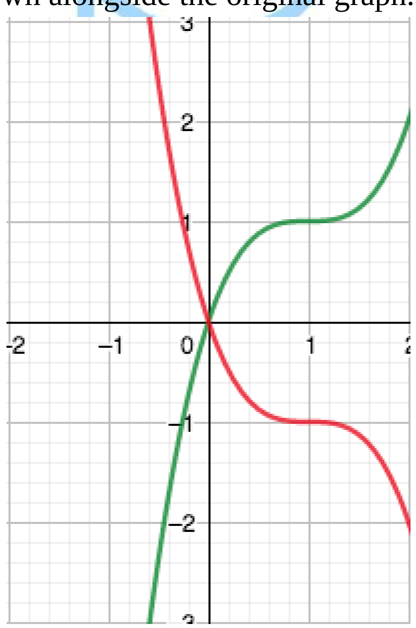


However, if we want to graph the function $y = f(x - 1) + 1$, the graph we obtain is shifted one unit to the right along the x-axis and moved up by one unit along the y-axis. This is what the resultant graph looks like:

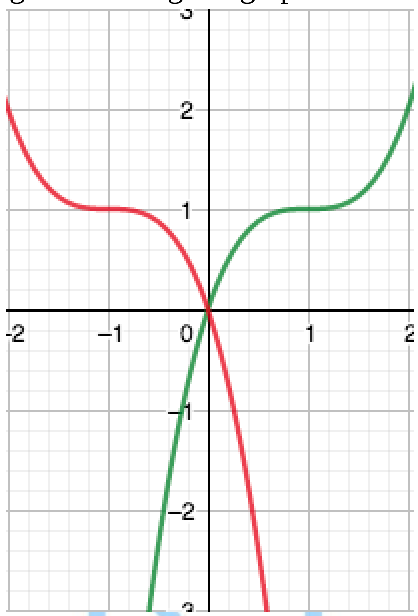


We can apply the same principle to all functions.

2. <u>Reflections in both axes:</u> $y = -f(x)$; $y = f(-x)$. $y = -f(x)$reflects the function f(x) over the x-axis. You essentially have to fold the graph over the x-axis. Let us consider the same example as mentioned earlier $y = (x - 1)^3 + 1$. The reflection of the graph in the x-axis is shown alongside the original graph.



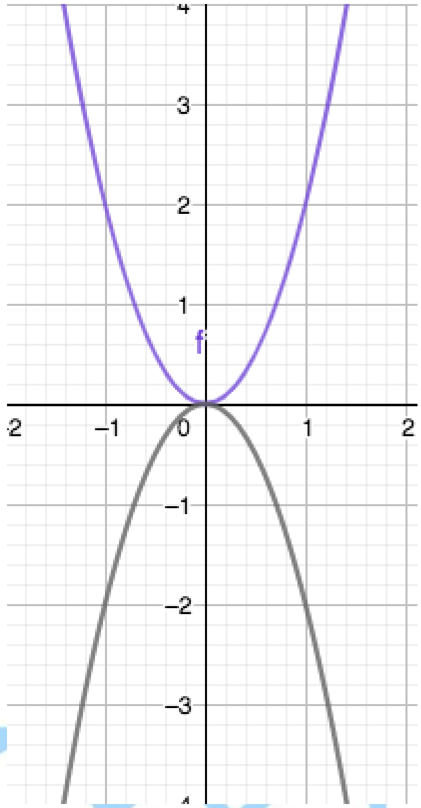
Now, let us look at $y = f(-x)$. $y = f(-x)$reflects the function f(x) over the y-axis. You essentially have to fold the graph over the y-axis. Using the same example, the reflection of the graph in the y-axis is shown alongside the original graph.



However, there are certain functions where f(x)=f(-x). These types of functions are known as even functions. A simple example of this is $f(x) = x^2$. In this function, if we substitute 'x' with '-x', we will get the same output function. Odd functions, on the other hand, are functions where f(-x) = f(x). An example for this is $f(x) = x^3$. If we substitute '-x' in 'x', we get $f(-x) = -x^3$which is equal to -f(x).

3. Dilations of functions: In the previous transformations, we only adjusted the 'position' of the graph of the function. In this type of transformation, we will actually change the shape of the graphs. A vertical stretch with scale factor 'p' is noted as y = pf(x). When p>1, the graph is vertically stretched, that is, the graph is pulled away from the x-axis. We find the coordinates of the graph by multiplying each of its y-coordinates by p. If 0<p<1, the graph is vertically compressed, that is, the graph is pulled towards the y-axis. Similar to earlier, we find the coordinates of the graph by multiplying each of its y-coordinates by p. If p<0, we first stretch the function by p and then reflect the graph across the x-axis. Essentially, the x-intercepts of the function will stay the same but the y-intercept will change. An example of is $y = -2x^2$. First, we pull the function of $f(x) = x^2$away from the x-axis by a scale factor of 2. That results in the graph shown below.

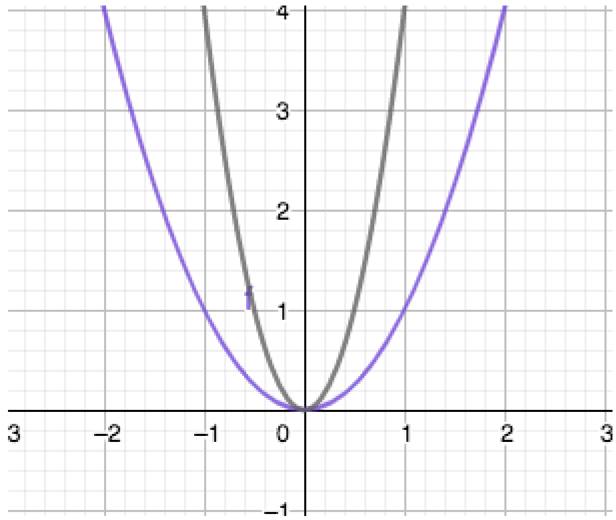
Then, we reflect this graph across the x-axis and we get this resulting graph.



Next, we are going to look at horizontal stretches, given by y = f(qx) where $\frac{1}{q}$ is the scale factor.

When q>1, the graph is horizontally stretched, that is, the graph is pulled away from the y-axis. We find the coordinates of the graph by dividing each of its y-coordinates by q. If 0<q<1, the graph is horizontally compressed, that is, the graph is pulled towards the x-axis. Similar to earlier, we find the coordinates of the graph by dividing each of its y-coordinates by q. If q<0, we first stretch the function by $\frac{1}{q}$ and then reflect the graph across the y-axis. Essentially, the x-intercepts will change, but the y-intercept will stay the same. Let us consider the function $f(x) = x^2$. Let us substitute 'x' with '2x'.

The graphs are illustrated below where f(x) is in blue and f(2x) is in grey.



In this particular case, as the function is an even one, the graph of f(-2x) is the same as the graph in blue.

Note- There is a particular sequence that must be followed when transforming functions as changing the order of transformations will create different functions. The sequence is as follows:

1. Start by looking for a horizontal stretch
2. Then check for any vertical stretch
3. Check for any reflections
4. Handle any vertical or horizontal transformations

Exercise:

Note down and graph the following functions after the following transformations.

1. $y = x^2$ after it has shifted to the right by 3 units, shifted up by 2 units and vertically stretched by a scale factor of 3.
2. $y = \frac{1}{x-2}$ after it has shifted to the left by 2, shifted up by 5 units, reflected in the x-axis and horizontally stretched by a scale factor of 0.5.

PART K

$$g(x) \geq f(x)$$

There are 2 methods through which we can determine the values of x for which $g(x) \geq f(x)$.

1. Solving algebraically: Equating the two equations and solving to obtain their points of intersection. Then, compare the functions at x-values that lie between the points of intersection to determine which function is greater between the points of intersection. Let us assume the graphs intersect at (m,n) and (p, q). If f(x)> g(x) between m and p, then g(x) is greater than f(x) for x>p and x<m.
2. Solving graphically. Graph the two equations and determine their points of intersection. Additionally, observe which graph is above the other at before, between and after their intersections.

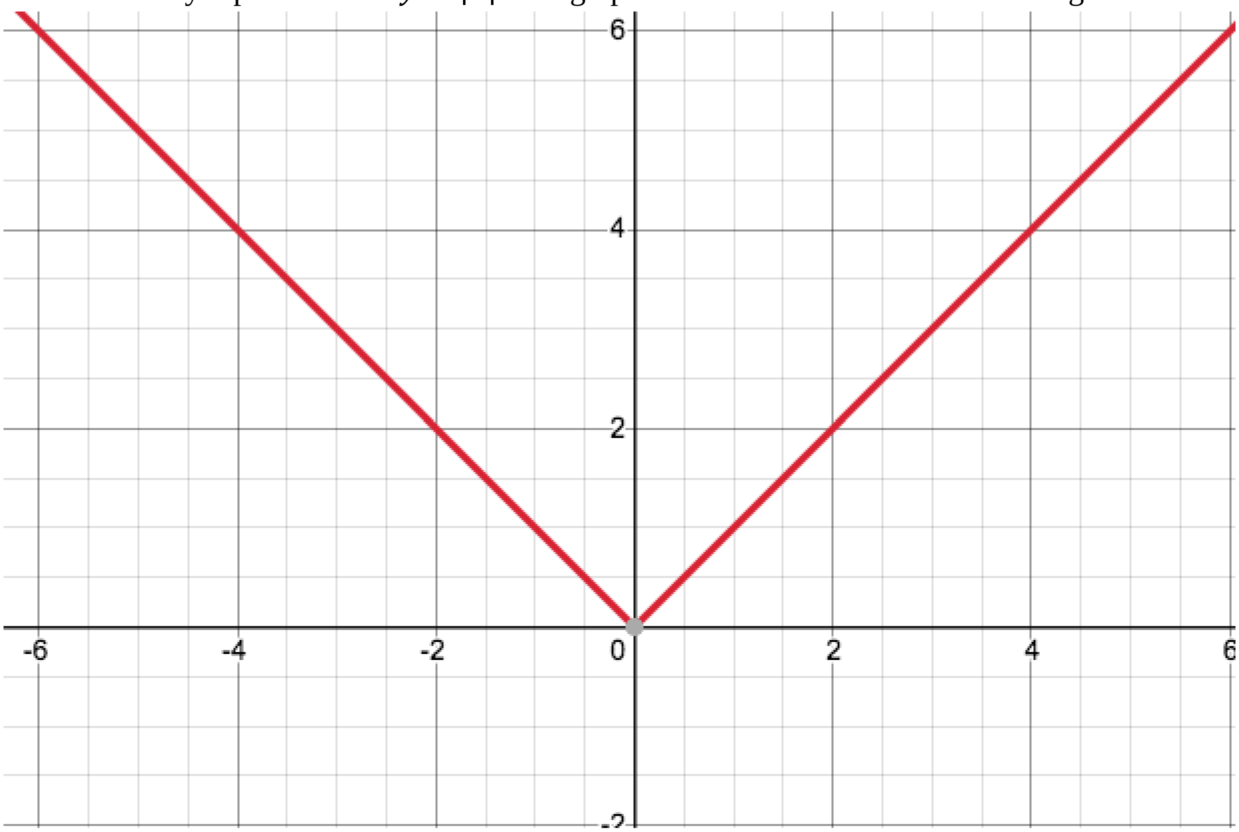
Exercise 2K:

1. Find the values of x for which $y = 3sin(x) + 2$is greater than or equal to $y = e^x + 5$.
2. Find the values of x for which $y = x^3 + 4x^2 - 3x + 1$ is greater than or equal to $y = x^2 + 5$.

PART L

Modulus and Piecewise Functions:

A modulus function is a function that has a positive output irrespective of the input. It is mathematically represented as $y = |x|$. The graph of the basic modulus function is given below.



This function can be noted as a piecewise function as well. A piecewise function is a function that is defined for a particular set of intervals. For the modulus function, the piecewise function is given by the following:

$$f(x) = \begin{cases} x, & if\, x \geq 0 \\ -x, & if\, x < 0 \end{cases}$$

Exercise:

Graph the following functions:

1. $f(x) = y = |x^2 - 3x - 4|$

2. $f(x)=\begin{cases}-2^x, & x<-4\\ -|x|, & -4\le x\le 0\\ 4-x^2, & x>0\end{cases}$

3. $w(x)=\begin{cases}|x-3|, & x<1\\ (x-1)^4, & x=1\\ \sqrt{4x}, & x>1\end{cases}$

4. $f(x) = y = |3(x-2)^3 - 1|$

PART M

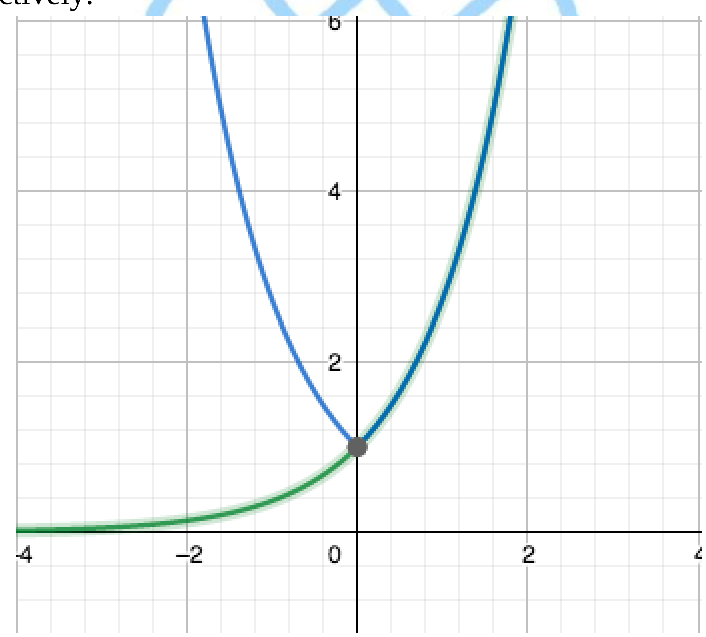
Advanced Transformations of Functions:

Some transformations of functions that are vital in Math AA HL are:

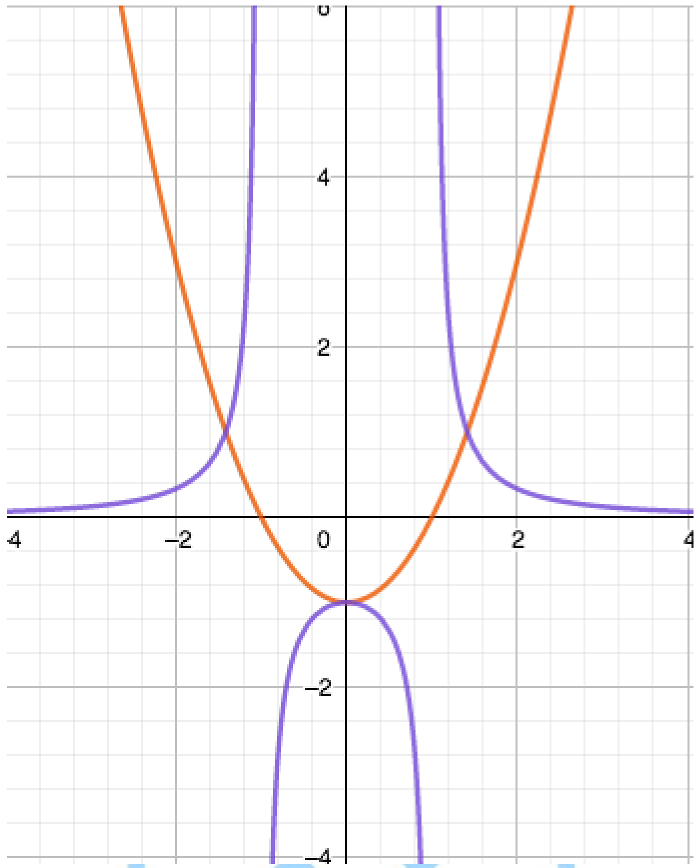
y = f(|x|), $y = \frac{1}{f(x)}$, y = $[f(x)]^2$

Let us look at the first one. y = f(|x|). This function reflects the graph to the right of the y-axis in the y-axis while ignoring the original left-hand side of the graph.

Eg- $f(x) = e^x$. $f(|x|) = e^{|x|}$. The original graph and the transformed graph are shown below, in green and blue respectively.



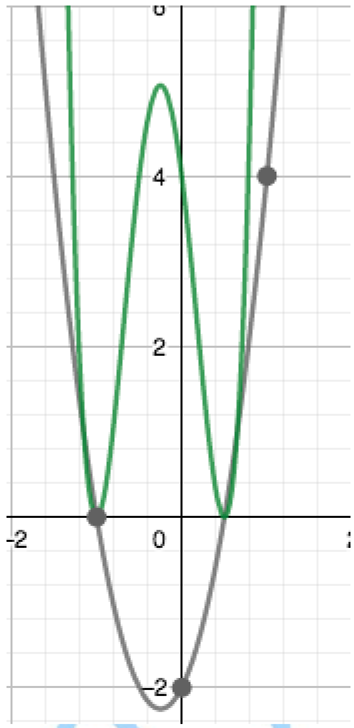
Next, is the graph $y = \frac{1}{f(x)}$. Here, we must take the reciprocal of each y-value of the original function. Let us consider a different example. $f(x) = x^2 - 1$.



The original function $f(x) = x^2 - 1$ is in orange and the transformed function $y = \frac{1}{f(x)}$ is in blue.

For these transformations, we must use our learnings from rational functions in order to find the asymptotes of the function and use any y-values of the original function to help understand the shape of this function.

Finally, let's look at $y = [f(x)]^2$. The output of this function is all positive as the square of any number is positive. Let us consider the function $f(x) = 4x^2 + 2x - 2$. $[f(x)]^2$ will be $(4x^2 + 2x - 2)^2$. The original and transformed function will be shown below in grey and green respectively.



Exercise:

Apply the 3 transformations discussed above on the functions given below.

1. $f(x) = \frac{1}{2x-3}$
2. $f(x) = \frac{x^2+6x+8}{x-3}$
3. $f(x) = e^{x-1} - 7$
4. $f(x) = x^3 - 4x^2 + 4x - 1$

PART N

Polynomial Functions:

What is a polynomial?

A **polynomial** is a function which can be written in the form

$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0 = \sum_{r=0}^{n} a_r x^r$ where $a_0, a_1, ..., a_n$ are constants, $a_n \neq 0$.

We say that:
- x is the **variable**
- a_0 is the **constant term**
- a_n is the **leading coefficient**
- a_r is the **coefficient of x^r** for $r = 0, 1, 2, ..., n$
- n is the **degree** of the polynomial, being the highest power of the variable
- $a_r x^r$ is the **term** of the polynomial with degree r.

A **real polynomial** $P(x)$ is a polynomial for which $a_r \in \mathbb{R}$, $r = 0, 1, 2, ..., n$.

What is the difference between the zeroes and the roots of a polynomial function?

A zero of a polynomial is a value of the variable, which makes the polynomial equal to zero. The roots of a polynomial equation are the solution to the equation.

Complex, Conjugate Roots Theorem:

This theorem states that complex roots are always found in pairs. Simply put, if we find, or are given, one complex root, we can easily obtain the other root. Similar to radical conjugates, the roots are normally of the form $a \pm bi$. The most important aspect that must be remembered when dealing with this theorem is that i^2 is equal to -1 as $i = \sqrt{-1}$. Additionally, in a quadratic equation, if one of the roots of the equation is a+bi, the other root must be a-bi, as the coefficients of x in the quadratic equation must be real.

Example:

3+i is a zero of $x^2 + ax + b$ where a and b are real numbers. Find a and b

As it is given that 3+i is one of the zeroes, we can easily deduce that the other zero is 3-i. To find the values of a and b, we use the expansion method. We multiply (x-3-i) by (x-3+i) and after expansion, we compare the coefficient of x to ax and the y-intercept to b and derive that they are equal.

Remainder and Factor Theorem:

The remainder theorem states that for a polynomial f(x), the remainder of f(x) upon division by x-c is f(c).

The factor theorem states that if we let f(x) be a polynomial such that f(c)=0 for some constant 'c'. Then x-c is a factor of f(x). Conversely, if x-c is a factor of f(x), then f(c)=0

Example:

(2x+1) and (x-2) are factors of $P(x) = 2x^4 + ax^3 + bx^2 + 18x + 8$. Find constants a and b and all zeros of P(x)

We substitute $x = -\frac{1}{2}$ and x=2 in the function p(x) and get two equations in terms of a and b. We then solve them simultaneously. Once we have obtained the values of a and b, we utilise them to divide the function by (2x+1) and (x-2) in order to get a quadratic. We do this using either long division or synthetic division.

Steps for synthetic division:

1. We need to find at least one factor of the function we are trying to simplify. This can be done by substituting a range of x-values in the function p(x), normally from -3<x<3.
2. Once we have found the factor, we draw a line on a paper and write the factor on the left hand side and the coefficients of x on the right, in descending order of the power it is

raised to.

3. Then, bring the leading coefficient (first number) straight down.
4. Multiply the number in the division box with the number you brought down and put the result in the next column
5. Add the two numbers together and write the result in the bottom of the row
6. Repeat steps 4 and 5 until you have reached the last term and write the final answer.
7. The final answer is made up of the numbers in the bottom row with the last number being the remainder and the remainder must be written as a fraction. The variables or x's start off one power less than the original denominator and go down one with each term.

Once we have the quadratic, we can use the quadratic formula to obtain the other roots.
Note- Complex conjugates can too be utilised with the factor theorem and having complex roots is always helpful as we automatically know two of the roots of the function.

<u>Sum and Product of the Roots:</u>

In simple terms:

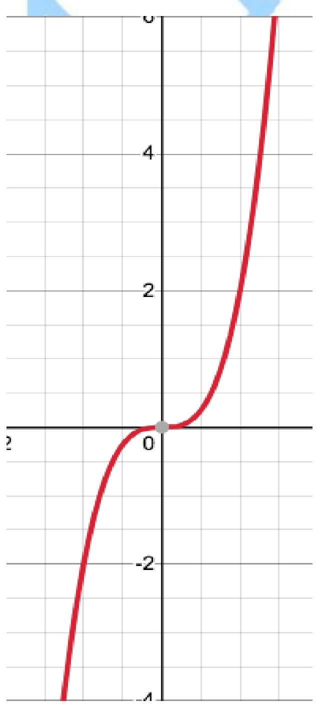
The sum of the roots of a polynomial function= $\frac{-coefficient\ of\ second\ highest\ power}{leading\ coefficient}$.

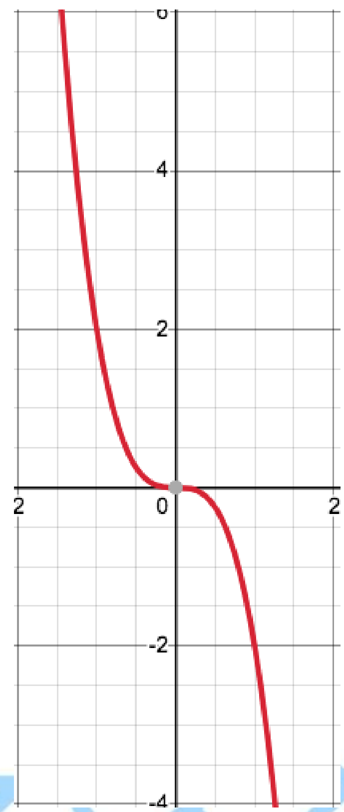
The product of the roots of a polynomial function= $\frac{(-1)^{degree\ of\ function}.constant}{leading\ coefficient}$.

<u>Graphing cubic and quartic functions:</u>

The first important thing that we must consider is the general shape of the graph of a cubic function. The graph of $y = ax^3$ has the shape given below when a>0

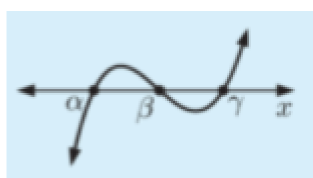
And when a<0, the graph of $y = ax^3$ has the shape given below.



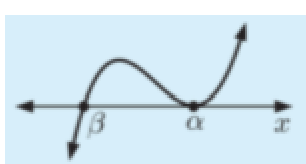


As a increases, the graph of y becomes steeper. Using our learnings from transformations, we recognise that the graph of $y = a(x-b)^3 + c$ is produced by translating the original graph of y by b units to the right/left and c units upwards/downwards. All cubic polynomials have at least one real zero and thus, must cut the x-axis at least once. Here is some additional information about the zeroes of cubic polynomials which will be immensely helpful in graphing them.

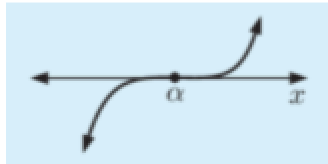
For a cubic of the form $P(x) = a(x-\alpha)(x-\beta)(x-\gamma)$where $\alpha, \beta, \gamma \in R$, the graph has three distinct x-intercepts corresponding to the three distinct roots $\alpha, \beta, and\ \gamma$.



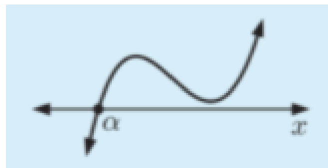
For a cubic of the form $P(x) = a(x-\alpha)^2(x-\beta)$where $\alpha, \beta \in R$,the graph touches the x-axis at α and β.



For a cubic of the form $P(x) = a(x-\alpha)^3, x \in R$,the graph has only one x-intercept, α. The graph is horizontal at this point.

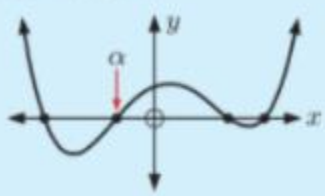


For a cubic in the form $P(x) = a(x-\alpha)(ax^2 + bx + c)$where the discriminant is negative, there is only one x-intercept, α.

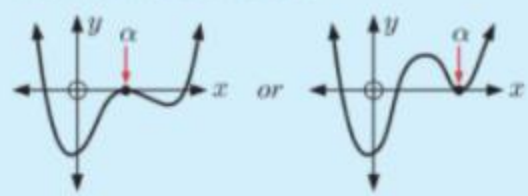


For quartic polynomials, $y = ax^4$.If a > 0, the graph opens upwards and if a<0, the graph opens downwards.
If a quartic is factored into real linear factors then:

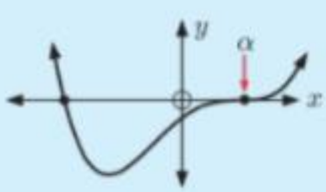
- for a **single factor** $(x-\alpha)$, the graph **cuts** the x-axis at α



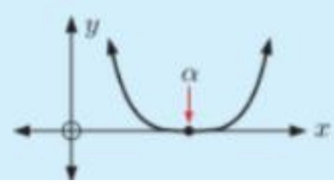
- for a **square factor** $(x-\alpha)^2$, the graph **touches** the x-axis at α



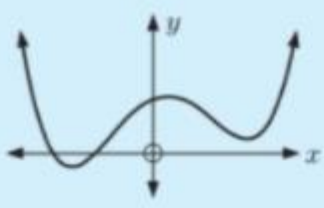
- for a **cubed factor** $(x-\alpha)^3$, the graph **cuts** the x-axis at α and is "flat" at α



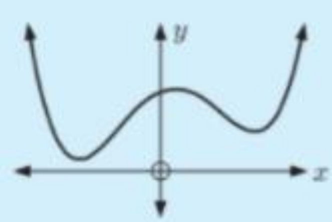
- for a **quadruple factor** $(x-\alpha)^4$, the graph **touches** the x-axis and is "flat" at that point.



- If a quartic with $a > 0$ has one real quadratic factor with $\Delta < 0$ we could have:



- If a quartic with $a > 0$ has two real quadratic factors both with $\Delta < 0$, the graph does not meet the x-axis at all.



Exercise:

1. Find all real quartic polynomials with zeros:
   a. -1, 2, $1 \pm \sqrt{5}$
   b. $3 \pm i, \pm\sqrt{13}$
   c. $\pm i\sqrt{5}, 2, 9$
2. Show that $x^4 + 16$ is divisible by two real quadratic factors that have the form $x^2 + ax + 4$ and $x^2 + bx + 4$, but cannot be divided into two real quadratic factors that have the form $x^2 + ax + 8$and $x^2 + bx + 2$.
3. When P(x) is divided by x^2-2x-3, the quotient is x^2+3x+2 and the remainder R(x) is unknown. When P(x) is divided by x-3, the remainder is 12. When P(x) is divided by x-2, the remainder is 80. Find R(x) in the form ax+b
4. 3+i is a root of $x^4 + ax^3 - 6x^2 - 4ax - b = 0$. Find a and b and deduce the other roots of the equation.
5. A real quartic polynomial has a leading coefficient of 3 and constant term 20. Its zeros exist in the form $2a \pm i$ and $5 \pm a$ where a is a real number. Find the possible values of a.
6. Find the equation of the quartic whose graph:
   a. Touches the x-axis at 2 and cuts the x-axis at -3 and $\frac{1}{4}$and passes through the point (-2, -14)
   b. Cuts the x-axis at $\pm\frac{2}{5}$and $\frac{1}{3}$and passes through the point (-3, 5)
   c. Passes through (-4, 14), (9,5), (-2,1) and (4,3)

IB Questions:

1. (Non-Calculator) The function g(x) is defined by g(x) = x^3 - $3x^2$ + 8x − 24 where x∈R
   (a) Find the remainder when g(x) is divided by
   (i) (x - 2)
   (ii) (x - 3).
   (b) Prove that g(x) has only one real zero.
   (c) Write down the transformation that will transform the graph of y = g(x) onto the graph of y = $8x^3$ - $12x^2$ + 16x - 24.
2. (Non-Calculator) Consider the following two functions:
$$b(x) = \frac{2x^2 + 3}{75}, x \geq 0.$$
$$c(x) = \frac{|3x - 4|}{10}, x\epsilon R$$
   (a) State the range of b(x) and c(x) and graph the two functions.
   (b) Find an expression for the composite function $(boc)(x)$ in the form $\frac{px^2+qx+r}{3750}$where p ,

q and r are integers.

(c) (i) Find an expression for the inverse term $f^{-1}(x)$.

(ii) State the domain and range of $f^{-1}(x)$ and graph the function.

3. (With Calculator) Find the set of values of x for which $|0.1x^2-2x+3|=\log_{10}(x)$
4. The functions p and q are defined by $p(x) = ax^2 + bx + c$ and $q(x) = d\sin(x) + ex + f, x\epsilon R$ where a, b, c, d, e, and f are real constants.
    a. Given that p(x) is an even function, show that b=0.
    b. Given that q(x) is an even function, find the value of f.
    c. The function r(x) is both even and odd, with domain R. Find h(x).
5. The function g(x) is of the form $g(x) = \frac{x+a}{bx+c}, x \text{ is not equal to } \frac{-c}{b}$.given that the graph of f has asymptotes $x = -4$ and $y = -2$, and that the point $(\frac{2}{3},1)$ lies on the graph, find the values of a , b and c .
6. (Non-Calculator)
    a. Sketch on the same axes the curve $y = \left|\frac{7}{x-4}\right|$ and the line y=x+2, clearly indicating any axes intercepts and any asymptotes.
    b. Find the exact solutions to the equation $x + 2 = \left|\frac{7}{x-4}\right|$
7. The quadratic equation $x^2 - 2kx + (k - 1) = 0$ has roots α and β such that $\alpha^2 + \beta^2 = 4$. Without solving the equation, find the possible values of the real number k.
8. In the quadratic equation $7x^2 - 8x + p = 0, p\epsilon Q$, one root is three times the other. Find the value of p.
9. Let r(x)= ln(x). The graph of r(x) is transformed into the graph of the function s by a translation of 3 units to the right and 2 units downwards, followed by a reflection in the x-axis. Find an expression for s(x), giving your answer as a single logarithm.
10. Find the values of k such that the equation $x^3 + x^2 - 3x + 1 = k$ has 3 distinct real solutions.
11. (With Calculator)The graph of $y = ln(x)$is transformed into the graph of $y = ln(2x + 1)$.
    a. Describe the two transformations that are required to do so
    b. Solve $ln(2x + 1) > 3sin(x), x\epsilon[0,10]$.
12. Consider the function $f(x) = \frac{1}{x^2+3x+2}, x\epsilon R, x \text{ is not equal to } -2 \text{ or } -1$.
    a. Express x^2+3x+2 in the form $(x+h)^2+k$
    b. Factorise x^2+3x+2
    c. Sketch the graph of f(x), indicating on it the equations of the asymptotes, the coordinates of the y-intercept and the local maximum.
    d. Show that $\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$
    e. Sketch the graph of f(|x|).

Topic 3: Geometry and Trigonometry

PART A

Distance, 3D Volume, Surface Area:

Distance between 2 points in 3-Dimensional Space:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Midpoint of 2 points in 3-Dimensional Space:

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2}\right)$$

Volume and Surface Area of 3-D Solids:

Square Pyramid:

$$Volume = \frac{1}{3}b^2h$$

$Surface\ Area = b^2 + 2bs$

Right Circular Cone:

$$Volume = \frac{1}{3}\pi r^2h$$

$$Surface\ Area = \pi r^2 + \pi rl$$

Sphere:

$$Volume = \frac{4}{3}\pi r^3$$

$$Surface\ Area = 4\pi r^2$$

Hemisphere:

$$Volume = \frac{2}{3}\pi r^3$$

$$Surface\ Area = 2\pi r^2$$

IB Questions:

A factory packages coconut water in cone-shaped containers with a base radius of 5.2 cm and a height of 13 cm.

Find the volume of one cone-shaped container. *[2 marks]*

Find the slant height of the cone-shaped container. *[2 marks]*

Show that the total surface area of the cone-shaped container is 314 cm^2, correct to three significant figures. *[3 marks]*

The factory designers are currently investigating whether a cone-shaped container can be replaced with a cylinder-shaped container with the same radius and the same total surface area.

Find the height, h, of this cylinder-shaped container. *[4 marks]*

The factory director wants to increase the volume of coconut water sold per container. *[4 marks]*

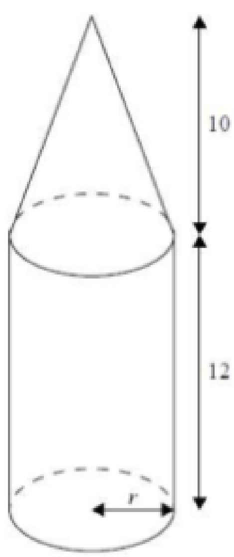
State whether or not they should replace the cone-shaped containers with cylinder-shaped containers. Justify your conclusion.

Julio is making a wooden pencil case in the shape of a large pencil. The pencil case consists of a cylinder attached to a cone, as shown.

The cylinder has a radius of r cm and a height of 12 cm.

The cone has a base radius of r cm and a height of 10 cm.

diagram not to scale

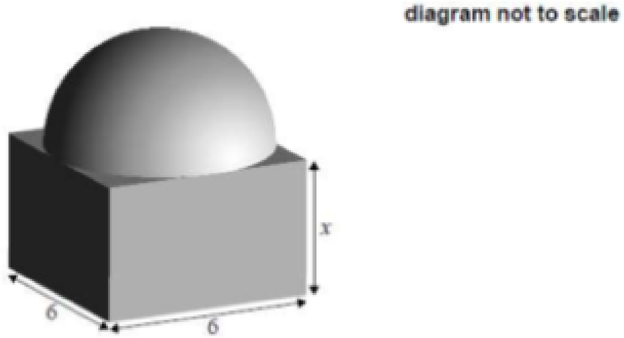


8a. Find an expression for the slant height of the cone **in terms of** $\boldsymbol{r}$. *[2 marks]*

8b. The total external surface area of the pencil case rounded to 3 significant figures is 570 cm^2. *[4 marks]*

Using your graphic display calculator, calculate the value of r.

A solid glass paperweight consists of a hemisphere of diameter 6 cm on top of a cuboid with a square base of length 6 cm, as shown in the diagram.



The height of the cuboid, x cm, is equal to the height of the hemisphere.

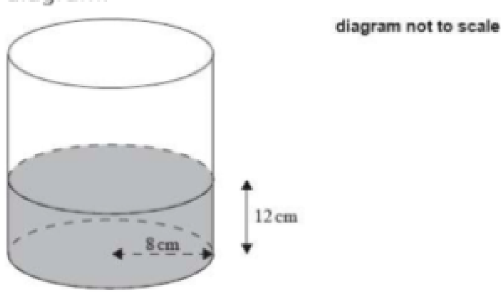
5a. Write down the value of x. *[1 mark]*

5b. Calculate the volume of the paperweight. *[3 marks]*

5c. 1 cm^3 of glass has a mass of 2.56 grams. *[2 marks]*

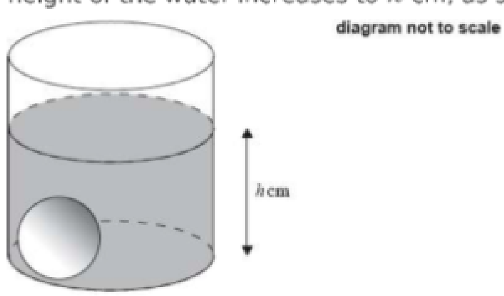
Calculate the mass, in grams, of the paperweight.

A cylindrical container with a radius of 8 cm is placed on a flat surface. The container is filled with water to a height of 12 cm, as shown in the following diagram.



11a. Find the volume of water in the container. *[2 marks]*

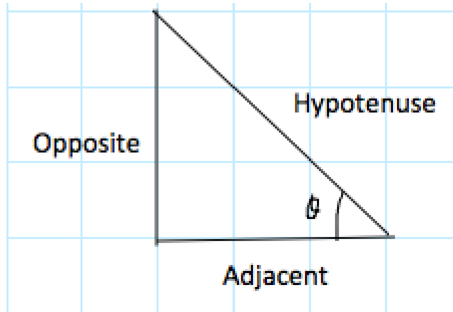
A heavy ball with a radius of 2.9 cm is dropped into the container. As a result, the height of the water increases to h cm, as shown in the following diagram.



11b. Find the value of h. *[4 marks]*

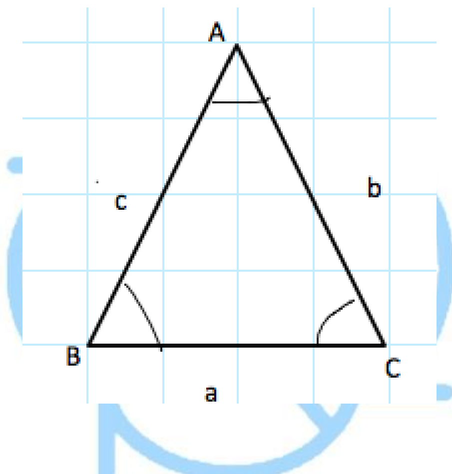
PART B

Right-Angled and Non Right-Angled Trigonometry:



$$\sin\theta = \frac{Opposite}{Hypotenuse}, \cos\theta = \frac{Adjacent}{Hypotenuse}, \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{Opposite}{Adjacent}$$

Sine Rule:



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Cosine Rule:

$c^2 = a^2 + b^2 - 2ab\cos C$;

Or

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.

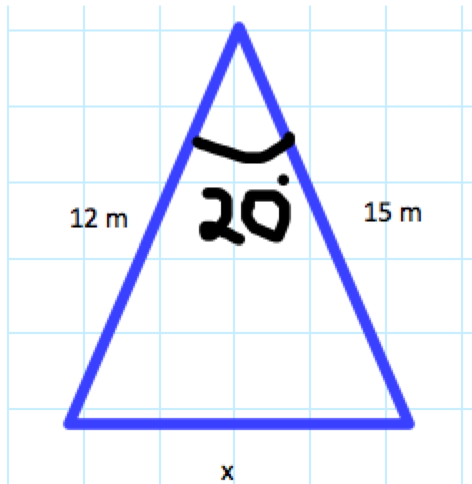
Area of a triangle using sine:

$$Area = \frac{1}{2}ab\sin C$$

Exercise 3B:

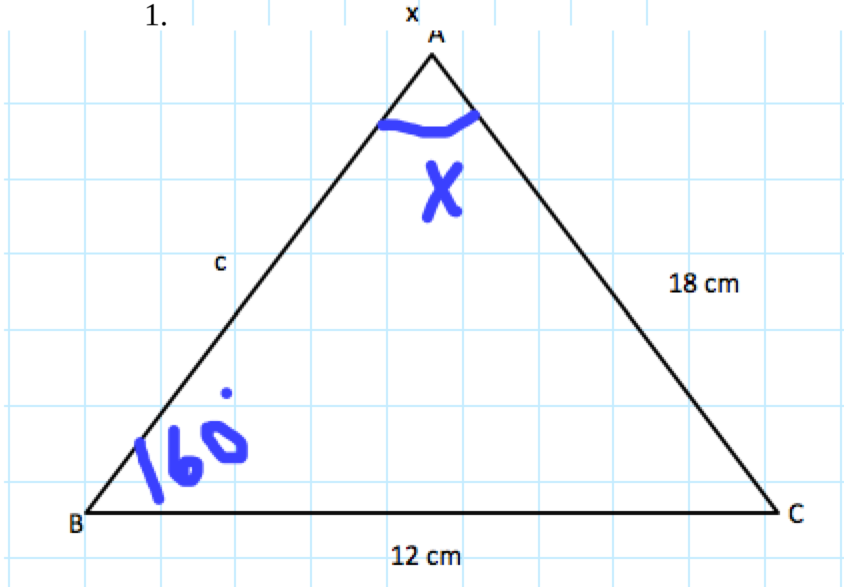
Find X:

1.



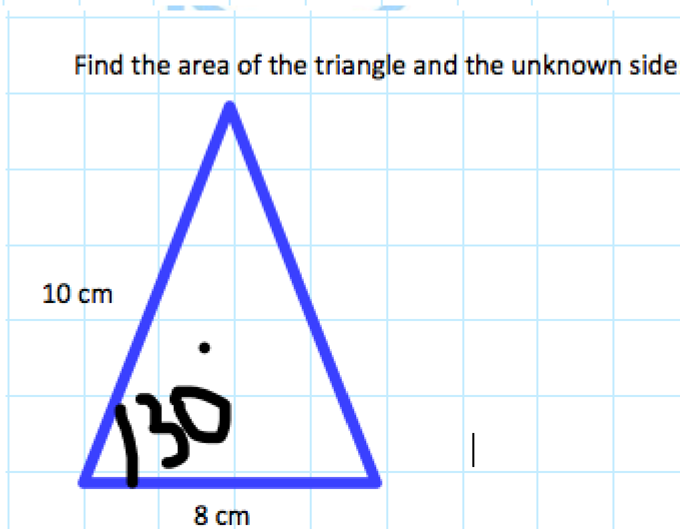
20°
12 m
15 m
x

2.



A
x
c
18 cm
160°
B
C
12 cm

3.



Find the area of the triangle and the unknown side:
10 cm
130°
8 cm

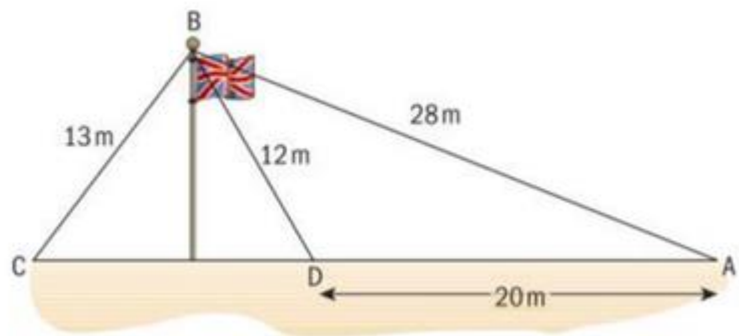
PART C

Applications of Right and Non Right-Angled Trigonometry:

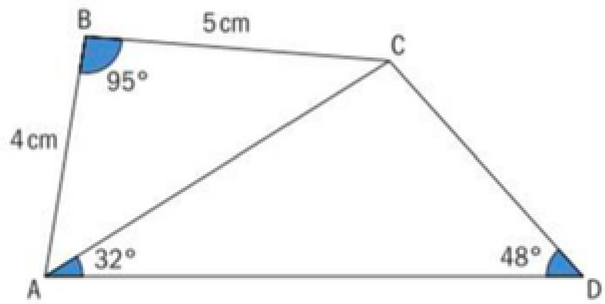
Now, we will look at the applications of Right and Non Right-Angled Trigonometry. The applications include bearings, angle of elevation and angle of depression.

Exercise:

1. A flagpole is held up by 3 wires as given in the diagram below.
   a. Calculate angle ADB
   b. Determine the area of ΔADB
   c. Determine the angle of elevation of the peak of the tower from point C
   d. Prove that ΔABC is not a right-angled triangle.



2. Quadrilateral ABCD is given below.
   a. Find the length of AC
   b. Find angle BAC
   c. Calculate the size of ACD
   d. Find the length of AD
   e. Determine the area of the quadrilateral ABCD.

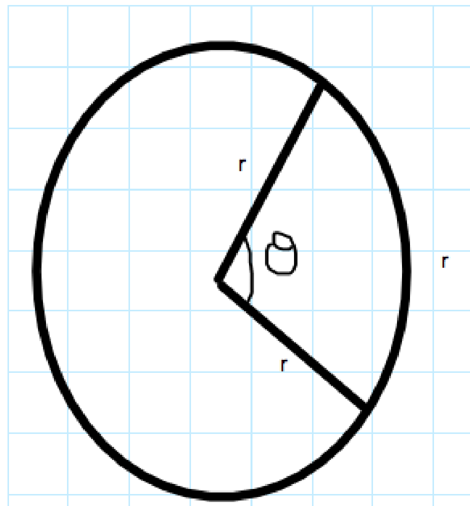


3. A ship leaves Jeju with a bearing of 030° and travels a distance of 250 km to Shanghai. At point B, the ship shifts direction with a bearing of 130°. It then travels a distance of 434 km to reach Nagasaki. Determine the bearing and distance for the ship to return from Nagasaki to Jeju.

PART D

Radian measure of angles, Length of an arc, and Area of a sector:

Radians are an alternative way of measuring angles.
One radian is equal to the angle that is formed when an arc opposite the angle has the same length as the radius of a circle. It is illustrated in a more simple manner below:



<u>Converting between radians and degrees:</u>

$$Radians = \frac{180}{Degrees}$$

Length of an arc:
In degrees, the length of an arc is equal to:

$$l = \frac{\theta}{360} \times 2\pi r$$

Converting to radians:

$$l = \theta r$$

Area of a sector:
In degrees, the area of a sector is equal to:

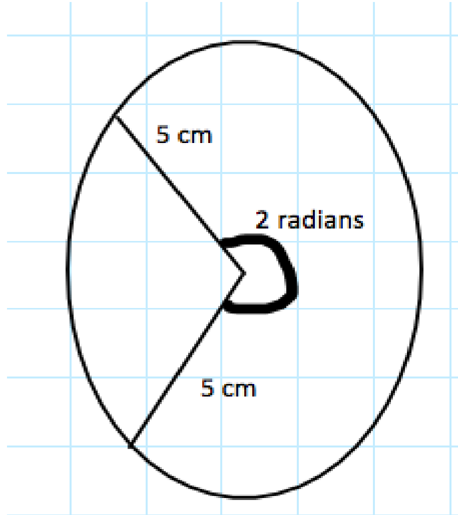
$$A = \frac{\theta}{360} \times \pi r^2$$

<u>Converting to radians:</u>

$$A = \theta r^2$$

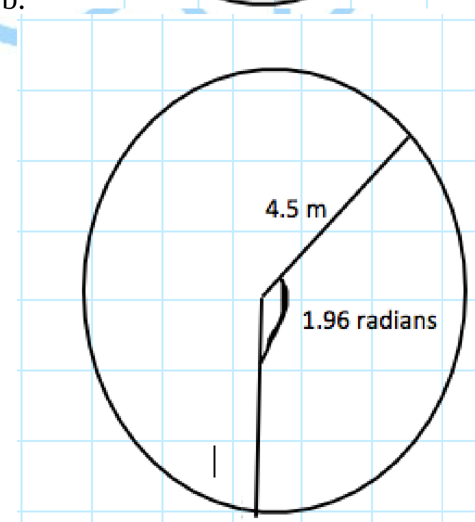
Exercise:

1. Find the length of the arc and the area of the sector:



a.

b.



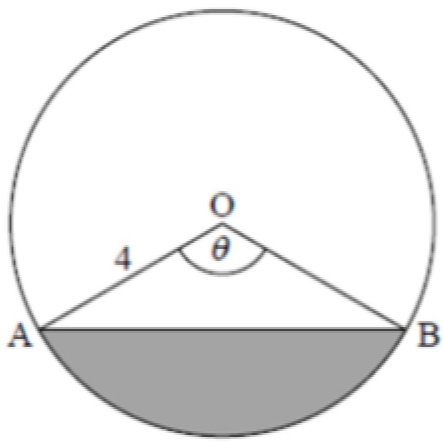
c.

IB Questions:

[3 marks]

The diagram shows a circle, centre O, with radius 4 cm. Points A and B lie on the circumference of the circle and AÔB = θ, where $0 \leq \theta \leq \pi$.

diagram not to scale



Find the area of the shaded region, in terms of θ.

[3 marks]

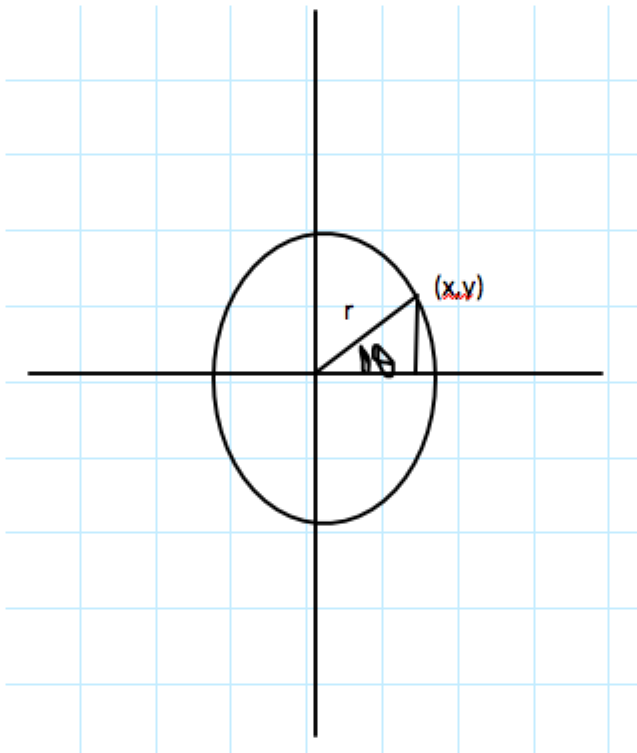
The area of the shaded region is 12 cm^2. Find the value of θ.

Find the value of r.

PART E

Unit Circle:

The equation of a unit circle is as follows: $x^2 + y^2 = 1$.



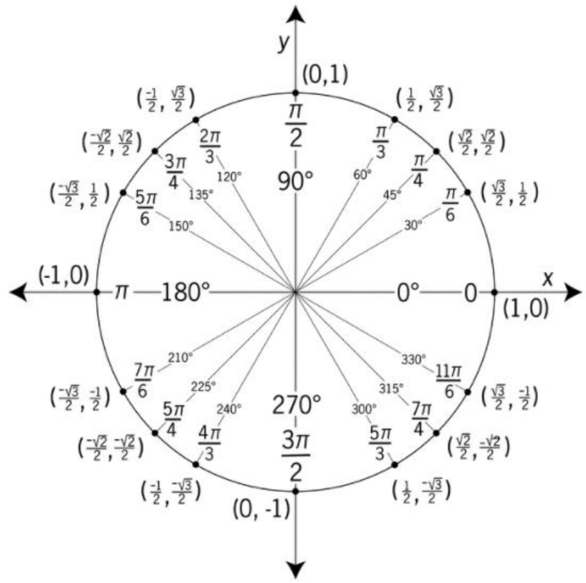
$$x = r\cos\theta$$
$$y = r\sin\theta$$

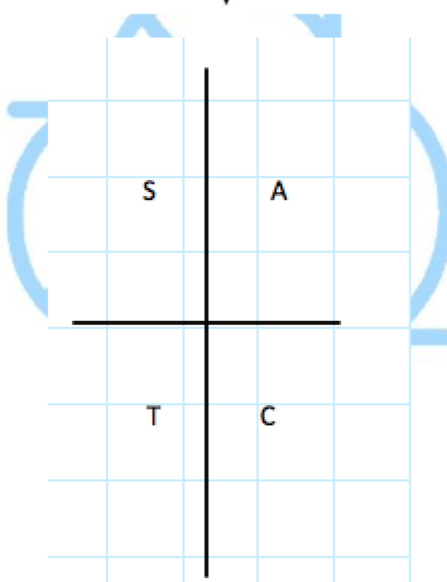
r =1

$$Hence, x = \cos\theta \text{ and } y = \sin\theta$$

Thus, we can deduce $(\cos\theta)^2 + (\sin\theta)^2 = 1$
Note: $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.
Thus, we can conclude that $-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$. This will be very important when we graph these functions.

Now, we can understand where a trigonometric function is positive.





A point in the first quadrant is given by (x, y), thus, both $cos\theta$and $sin\theta$ will be positive and as, tan$\theta$is $\frac{sin\theta}{cos\theta}$, it too is positive.

A point in the second quadrant is given by (-x, y), thus, while $sin\theta$ is positive, both $cos\theta$ and tan$\theta$are negative.

A point in the third quadrant is given by (-x, -y), thus, both $cos\theta$and $sin\theta$ will be negative and as, tan$\theta$is $\frac{sin\theta}{cos\theta}$, it is positive.

A point in the second quadrant is given by (x, -y), thus, while $cos\theta$ is positive, both $sin\theta$ and tan$\theta$are negative.

This concept can be remembered by a simple phrase: All Silver Tea Cups.

We need to know the values of the unit circle. The diagram below illustrates all the important ones.

While this may seem confusing, there is an easier method. We only need to learn the sine and cosine values in the first quadrant and use 'All Silver Tea Cups' and the relation tanθ= $\frac{sin\theta}{cos\theta}$to get everything else.

Necessary sine and cosine values:

- $cos\ 0 = sin\ 90 = sin\frac{\pi}{2} = 1$
- $sin\ 0 = cos\ 90 = cos\frac{\pi}{2} = 0$
- $sin\ 30 = sin\frac{\pi}{6} = \frac{1}{2}$. This is also equal to cos60 and cos$\frac{\pi}{3}$.
- $cos\ 30 = cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. This is also equal to sin60 and sin$\frac{\pi}{3}$.
- $sin\ 45 = sin\frac{\pi}{4} = cos\ 45 = cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Other important relations:

1. $sec\theta\ = \frac{1}{cos\theta}$
2. $cosec\theta\ = \frac{1}{sin\theta}$
3. $cot\theta\ = \frac{1}{tan\theta} = \frac{cos\theta}{sin\theta}$

<u>Trigonometric Identities:</u>

Supplementary Angles:
1. $cos(180 - \theta) = cos(\pi - \theta) = -cos\theta$
2. $sin(180 - \theta) = sin(\pi - \theta) = sin\theta$
3. $tan(180 - \theta) = tan(\pi - \theta) = -tan\theta$

Negative Angles:
1. $cos(-\theta) = cos\theta$
2. $sin(-\theta) = sin\theta$
3. $tan(-\theta) = -tan\theta$

Complementary Angles:
1. $cos(90 - \theta) = cos(\frac{\pi}{2} - \theta) = sin\theta$
2. $sin(90 - \theta) = sin(\frac{\pi}{2} - \theta) = cos\theta$
3. $tan(90 - \theta) = tan(\frac{\pi}{2} - \theta) = \frac{1}{tan\theta}$

These identities are very important for simplifying various sums in the IB Syllabus and we will solve some of them together.

Example:

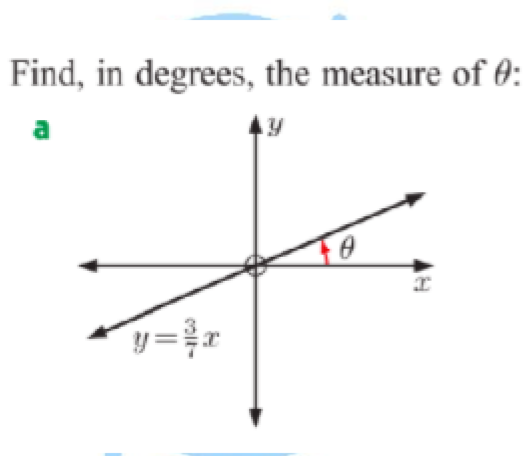
Evaluate $(sin\frac{2\pi}{3})^2 - (tan\frac{7\pi}{4})^2$.

We know that $sin(\pi - \theta) = sin\theta$. Thus $sin\frac{2\pi}{3} = sin(\pi - \frac{2\pi}{3}) = sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. Thus, its square is $\frac{3}{4}$. We can easily deduce that $tan\frac{\pi}{4} = 1$. $\frac{7\pi}{4}$lies in the 4th quadrant and the tan function is negative in that quadrant. Thus, $tan\frac{7\pi}{4} = -1$. Hence, its square is 1. Therefore, the above expression simplifies to:

$\frac{3}{4} - 1 = \frac{-1}{4}$.

Note- The equation of a straight line through the origin is y = xtanθ, where θ is the angle formed between the line and positive x-axis.

Exercise 3E:

1. 

2. Solve:
   a. $4(sinx)^2 - 3 = 0, 0 \le x \le 2\pi$
   b. $2(tanx)^2$-5=0, -$2\pi \le x \le 2\pi$
3. If $sinx = -\frac{3}{5}, \pi \le x \le \frac{3}{2}\pi$, find (hint, draw a right angled triangle to help yourself out)
   a. cosx
   b. tanx
   c. cos(π-x)
   d. tan(2π-x)
   e. cotx
   f. secx
   g. cosecx
   h. cosec($\frac{\pi}{2}$-x)
   i. sec($\frac{\pi}{2}$+x)

PART F

Trigonometric Identities and Double Angle Formula:

Important Trigonometric Identities:

1. $(cos\theta)^2 + (sin\theta)^2 = 1$
2. $(tan\theta)^2 + 1 = (sec\theta)^2$
3. $(cot\theta)^2 + 1 = (cosec\theta)^2$

Double Angle Formulae:

1. $sin2x = 2sinxcosx$
2. $cos2x = (cosx)^2 - (sinx)^2 = 2(cosx)^2 - 1 = 1 - 2(sinx)^2$
3. $tan2x = \frac{2tanx}{1-(tanx)^2}$

Exercise:

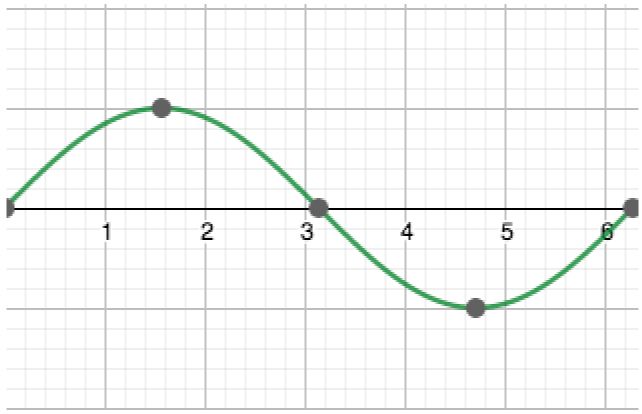
1. If x is acute and $cos2x = \frac{-7}{9}$, find
   a. Cosx
   b. Sinx
   c. Tanx
2. Find the exact value of tanA if $tan2A = \frac{21}{20}$and A is obtuse.
3. Prove that:
   a. $\frac{sin2x}{1-cos2x} = cotx$
   b. $(cos\theta)^4 - (sin\theta)^4 = cos2\theta$
   c. $cosec2\theta = tan\theta + cot2\theta$
   d. $\frac{sin2A}{sinA} - \frac{cos2A}{cosA} = secA$
4. Solve for x, $0 \le x \le 2\pi$
   a. $sin2x - cosx = 0$
   b. $cos2x - sinx = 0$
   c. $cos4x = cos2x$
   d. $2(sinx)^2 = 3cosx$
   e. $3(tanx)^2 - 13tanx + 4 = 0.(Hint, think\ back\ to\ 2^{2x}\ type\ sums.)$

PART G

Graphing and Modelling Trigonometric Functions:

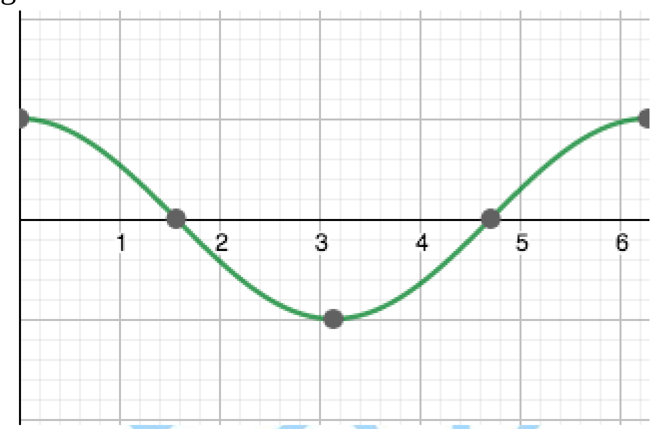
Graphing Trigonometric Functions:

The graph of sin(x) is given below for $0 \le x \le 2\pi$:



As sin(x) is a periodic function, the function exists in a similar manner for $x\epsilon R$.

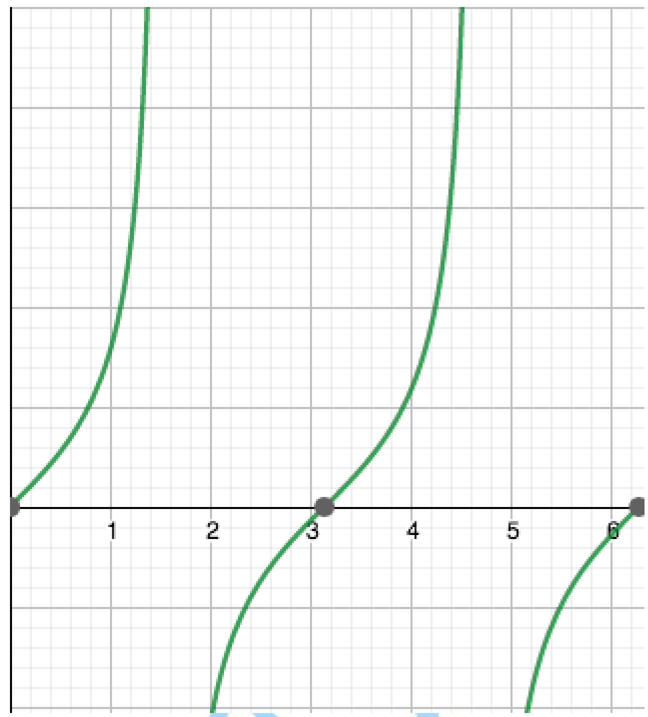
The graph of cos(x) is given below for $0 \leq x \leq 2\pi$



As cos(x) is a periodic function, the function exists in a similar manner for $x\epsilon R$.

However, tanθ= $\frac{sin\theta}{cos\theta}$. Thus, the function has a vertical asymptote (discussed in topic 2) whenever cos$\theta$is equal to zero, which is at $x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \ldots$.Essentially, there are vertical asymptotes at $x = \frac{\pi}{2}(2n + 1), where\ n\epsilon Z.$

Thus, the graph of tan$\theta$is given by the following for$0 \leq x \leq 2\pi, x \neq \frac{\pi}{2}(2n + 1)$:



The equation for a general sine function is given by the following:
f(x) = asin(b(x + c)) + d, where a is the amplitude of the function, b is the period, c is the horizontal translation and d is the vertical translation. The amplitude is the difference between the mean position of the graph and its highest/lowest point. The period of a trigonometric function is the minimum time it takes for a function to return from its mean position after reaching a maximum and a minimum. The period of the most simple sine graph, y=sinx, is 2π. Here, B=1, thus, we can conclude that $Period = \frac{2\pi}{B}$.Similar to a sine function, the period of a cosine function is $\frac{2\pi}{B}$and its amplitude is the difference between the mean position of the graph and its highest/lowest point. Additionally, the range of basic sine and cosine functions is $-1 \leq y \leq 1$. However, the range of tan function is $y\epsilon R$. Moreover, the period of the most simple tan function is π. Thus, its period in general is given by $\frac{\pi}{B}$.”à” for a tan function is not the amplitude, rather, it is the vertical stretch of the function.

<u>**Note:**</u>- We can convert sine and cosine graphs to each other very easily. In the most basic sine function, it can be transformed to a cosine function by a horizontal translation by $\frac{\pi}{2}$units to the left.

<u>Modelling Trigonometric Functions:</u>

Example:

1. Given below is a table that has recorded the average monthly temperatures for a city in India.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	14	15	18	21	25	27	26	24	20	18	16

   a. Plot the given data on a graph, taking the months as x-values (For example, Jan=1, Feb=2 and so on)
   b. This data can be modelled by a trigonometric function in the form T= asin(b(x + c)) + d. Without using technology, estimate the values of
      i. b
      ii. a
      iii. d
      iv. c
   c. Check how well your model fits the data using technology.

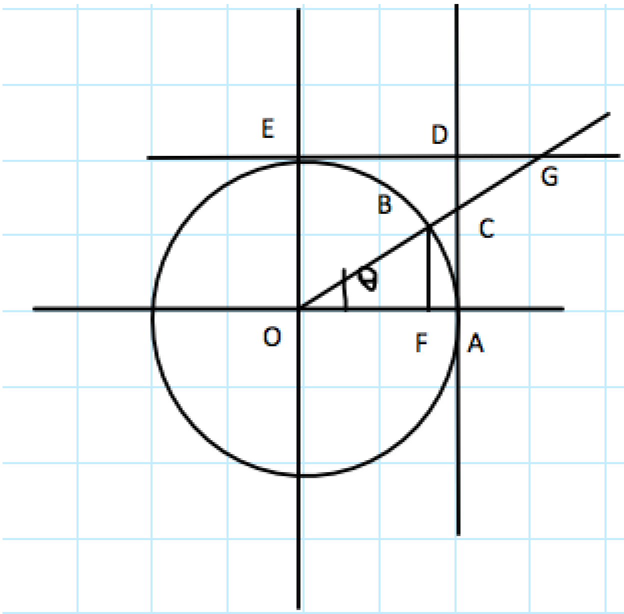
The first subpart is quite simple and just requires attention to detail while graphing. Next, roughly connect the points in a form resembling a trigonometric function and then analyse the graph to roughly estimate the period of the graph. To determine the amplitude a, we must look at the maximum and minimum temperature. $Amplitude = \frac{maximum-minimum}{2}$.The vertical translation is given by $d = \frac{maximum+minimum}{2}$. Finally, with only one variable left (c), we can simply use substitution to get its value.

Exercise:

1. For the function $y = 3sin(\frac{x}{4}) + 2$, find the following:
   a. Amplitude
   b. Period
   c. Range
2. Sketch the following functions for $0 \leq x \leq 3\pi$:
   a. $y = 2cos(x + \frac{\pi}{3}\ ) - 1$
   b. $y = sin2x + 5$
   c. $y = 3tan(2x - \pi) - 2$
3. State the transformations which transform:
   a. $y = sin\ x\ to\ y = sin(2(x - \frac{\pi}{4}\ ) + 2$
   b. $y = tan\ x\ to\ y = \frac{1}{5}tan(\frac{1}{2}(x + \pi)) + \frac{2}{5}$
4. On a June day in London, the maximum temperature is 18.7 °C occurs at 1:30 pm and the minimum temperature is 4.5°C. Suggest a cosine function that can model the temperature for that day, where T is the temperature and t is the hours after midnight.

PART H

Reciprocal Trigonometric Functions and Inverse Trigonometric Functions:

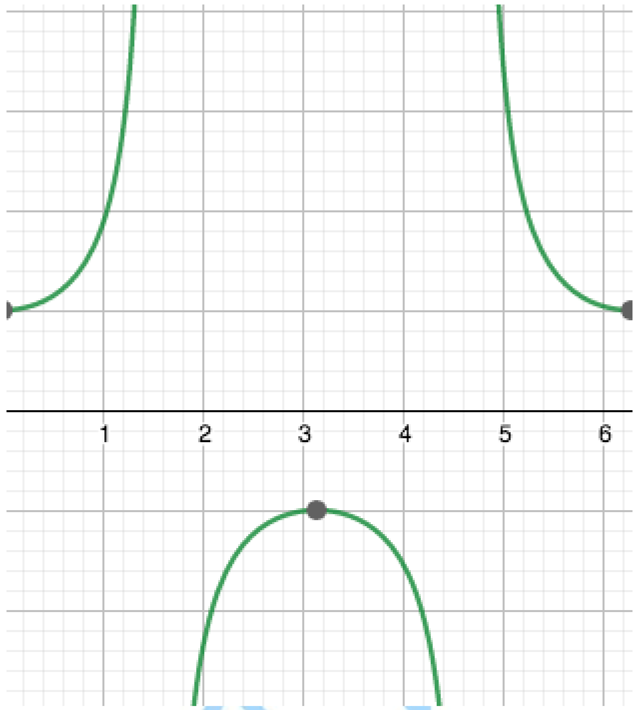


Thus, from the unit circle:

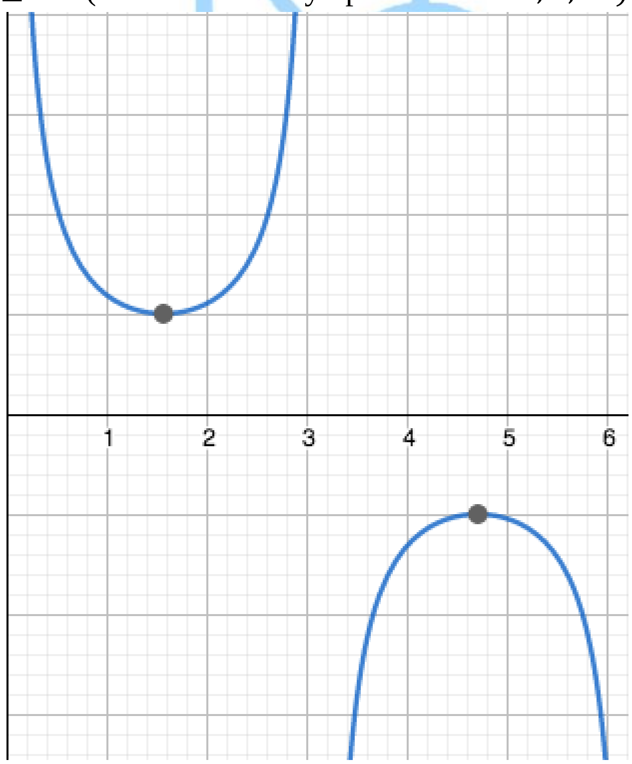
1. $sin\theta = BF$
2. $cos\theta = OF$
3. $tan\theta = AC$
4. $sec\theta = OC$
5. $cosec\theta = OG$
6. $cot\theta = EG$

<u>Sketching reciprocal functions:</u>

$y = sec\theta$ for $0 \leq \theta \leq 2\pi$: (It has vertical asymptotes at $x = \frac{\pi}{2}, \frac{3\pi}{2}$



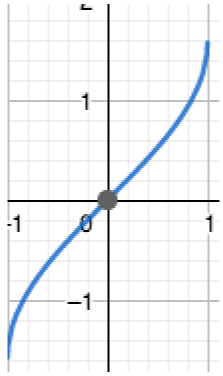
$y = cosec\theta \ for \ 0 \leq \theta \leq 2\pi$: (It has vertical asymptotes at $x = 0, \pi, 2\pi$)



We can easily deduce the graph of $y = cot\theta \ for \ 0 \leq \theta \leq 2\pi$, using our learning from $y = \frac{1}{f(x)}$ in advanced transformations in functions.

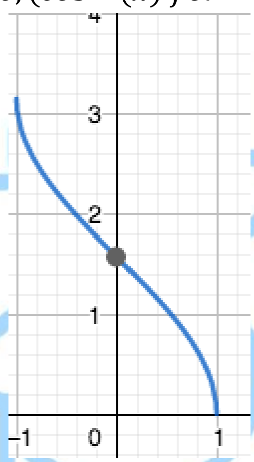
<u>Sketching inverse trigonometric functions:</u>

Let us first start with $y = arcsinx\ (sin^{-1}(x)\ for\ -1 \leq x \leq 1$



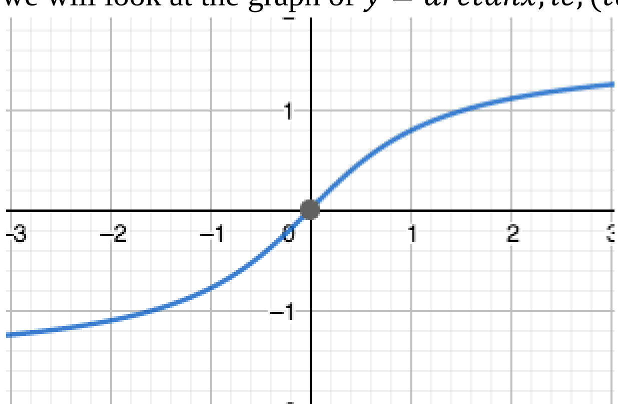
Domain$\epsilon[-1,1]$
Range$\epsilon\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Now, let us move onto $y = arccosx, ie, (cos^{-1}(x)\ for\ -1 \leq x \leq 1$



Domain $\epsilon[-1,1]$
Range$\epsilon[0,\pi]$

Finally, we will look at the graph of $y = arctanx, ie, (tan^{-1}(x)$



Domain: $D\epsilon R$
Range: $R\epsilon]-\frac{\pi}{2},\frac{\pi}{2}[$

Exercise:

1. Simplify:
    a. $\arccos(\cos\frac{\pi}{3})$
    b. $\arcsin(\sin-\frac{\pi}{4})$
    c. $\arctan(\tan\frac{4\pi}{3})$
2. Find the other 5 trigonometric ratios if
    a. $sin(x) = -\frac{2}{3}, \frac{3\pi}{2} < x < 2\pi$
    b. $tan(x) = \frac{5}{3}, \pi < x < \frac{3\pi}{2}$
    c. $sec(x) = -\frac{3}{1}, \frac{\pi}{2} < x < \pi$

PART I

Compound Angle Identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

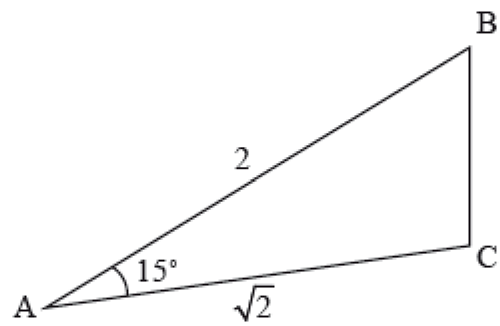
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Exercise:

1. Determine exactly the tangent of the acute angle between 2 lines with gradients $\frac{5}{3}$and $\frac{1}{7}$
2. If $\sqrt{2}sinx + cosx = ksin(x-a), for\ k > 0\ and\ 0 < a < 2\pi$.
3. We know that sinB=cosA+cosC. Prove that triangle ABC is right-angled.
4. If $tan(A-B) = \frac{6}{7}$and $tanB = \frac{3}{4}$, find the exact value of tanA.

IB Questions:

1. Prove that $\frac{cosA+sinA}{cosA-sinA} = sec2A + tan2A$
2. In triangle ABC, $3sinB + 4cosC = 6$and $4sinC + 3cosB = 1$
    a. Prove that $sin(B+C) = \frac{1}{2}$
    b. Show that angle CAB can have only one value
3. The diagram below illustrates the triangle ABC where AB=2, $AC = \sqrt{2}$,and angle BAC=15°



a. Expand and simplify $(1-\sqrt{3})^2$
b. By writing 15° as 60°-45°, determine the value of cos(15°)
c. Determine the value of BC in the form $a+\sqrt{b}$, where a, b ∈ Z

4. The first 3 terms of a geometric sequence are sinx, sin2x, 4sinx(cosx)^2, $-\frac{\pi}{2}<x<\frac{\pi}{2}$.
   a. Find the common ratio r
   b. Find the set of values of x for which geometric series sinx+sin2x+ 4sinx(cosx)^2+.. converges
   c. Let $x = arccos\left(\frac{1}{4}\right), x > 0$. Show that the sum to infinity of this series is $\frac{\sqrt{15}}{2}$.
5. Obtain all the solutions for the equation $tanx + tan2x = 0\ for\ 0 < x < 360°$
6. Given that $\frac{\pi}{2} < x < \pi$ and $cosx = -\frac{3}{4}$,find the value of sin2x.
7. In triangle ABC, angle ABC=90°, $AC = \sqrt{2}, AB = BC + 1$.
   a. Prove that $cosA - sinA = \frac{1}{\sqrt{2}}$
   b. By squaring both sides of the equation in part a, solve the equation to find the angles in the triangle
   c. Use Pythagoras theorem in triangle ABC to get BC, and hence, prove that $sinA = \frac{\sqrt{6}-\sqrt{2}}{4}$
   d. Hence, or otherwise, calculate the length of the perpendicular from B to [AC]
8. (With Calculator)
   a. Solve the equation $3(cosx)^2 - 8cosx + 4 = 0, where 0 < x < 180°$, expressing your answer to the nearest degree
   b. Determine the exact value of sec x that satisfy the equation $3(secx)^4 - 8(secx)^2 + 4 = 0,$
9. (Without Calculator)
   a. Sketch the graphs of $y = cosx$and $y = 3cos2x - 1\ for\ the\ domain\ 0 \leq x \leq \frac{\pi}{2}$.
   b. Find the x-coordinates of the point of intersection of the two graphs for the same domain given above.

Topic 4: Calculus

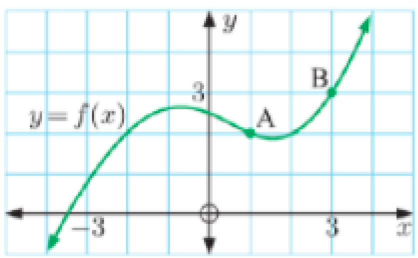
PART A

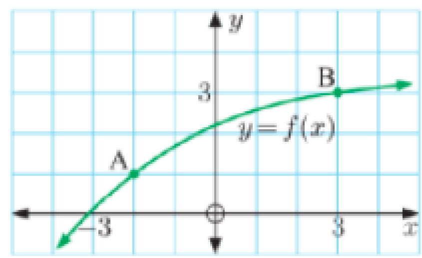
Differentiation:

Finding the derivative of a particular function means finding the rate of change of one variable with respect to the other. The rate of change can be defined as the average rate of change, or the instantaneous rate of change.

Note:- The slope of an equation is also known as the rate of change of the dependent variable with respect to the independent variable.

Using the examples below we will understand what exactly this means.

a. 

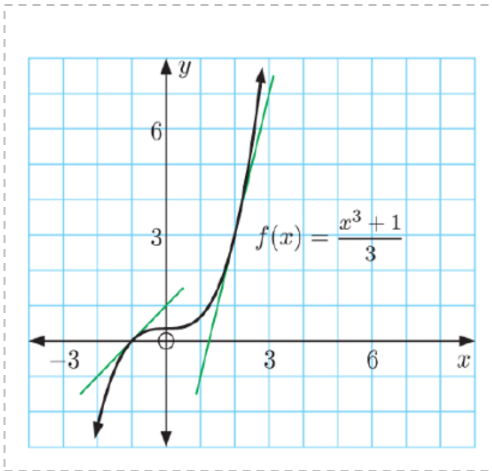
b. 

To find the average rate of change from A to B for example in question (a) we can find the slope of the line using the two points. A(1,2) and B(3,3).

Slope = $\frac{3-2}{3-1} = \frac{1}{2}$; Another way to look at it would be that we rise 1 up and run 2 to the right. This gives us the rate of change from A to B. Similarly, in the (b) we can use the same logic to calculate the rate of change from A(-2,1) to B(3,3). It would be :-

Slope:- $\frac{3-1}{3-(-2)} = \frac{2}{5}$

Finding the instantaneous rate of change, would mean finding the rate of change of ‘y’ with respect to ‘x’ at that particular x value. The example below illustrated the same.



2a. To find the instantaneous rate of change at $x = -1$, we draw a tangent at that point, and find its slope. By choosing two suitable points (1,-1) and (3,7) we can find the slope.

Slope:- $\frac{7-(-1)}{3-1} = \frac{8}{2} = 4$To find the instantaneous rate of change at $x = 2$, we find the slope of the tangent using two suitable points (0,1) and (-2,-1).

Slope:-$\frac{-1-1}{-2} = 1$

Derivatives can be found using the first principle's method as well. The first principle's method suggests that :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let us take an example to prove the following.

$$Let \quad f(x) = x^2$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2hx - x^2 + h^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h\,(2x + h)}{h}$$

$$f'(x) = \lim_{h \to 0} 2x + h$$

$$f'(x) = 2x + 0 = 2x$$

Therefore, the derivative of $x^2$is $2x$. For now let us use the concept of limits, which will be explained in detail at a later stage of the guide.

Usually, in order to understand derivatives fully it is imperative to understand that the derivative of a function can be used to find the gradient of the tangent to any curve at that particular point. For example, find the gradient of the tangent to:

a) $f(x) = x^3$at the point (2,8)

The following is solved by first, finding the derivative of the function and then plugging in the value of 'x' to find the gradient of the tangent at x = 2. So, using the first principle's method:

$$f(x) = x^3$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3h^2x + h^3 - x^3}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$f'(x) = \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$f'(x) = 3x^2 + 3x(0) + 0 = 3x^2$$

Now that we know that $f'(x) = 3x^2$,we can substitute 2 in place of 'x' to get $f'(2)$= 3(4) = 12. The gradient of the tangent at x = 2 is 12.

Q. Using the first principle's method find the gradient of the tangent to :

a) $f(x) = x^3 + x$, at (1,2)
b) $f(x) = \frac{4}{x} + 3$, at (2,5)

After we perform the same calculation over and over again for similar functions we can generalise a set of rules for differentiation.

The Derivative of x^n: $f'(x) = nx^{n-1}$

Below are some of the many common derivatives of different types of functions.

Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$
Derivative of e^x	$f(x) = e^x \Rightarrow f'(x) = e^x$
Derivative of $\ln x$	$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

Standard derivatives	
$\tan x$	$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$
$\sec x$	$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$
$\operatorname{cosec} x$	$f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x \cot x$
$\cot x$	$f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$
a^x	$f(x) = a^x \Rightarrow f'(x) = a^x (\ln a)$
$\log_a x$	$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$
$\arcsin x$	$f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$

Q. Suppose $y = \frac{c}{\sqrt{1+dx}}$ where c and d are constants. When $c = 3, d = 1$ and $\frac{dy}{dx} = -\frac{1}{8}$, find c and d.

Q. Suppose $f(x) = 3(cx - \frac{d}{x})^3$. Given that $f(1.5) = 3$ and $f'(1.5) = 30$, find c and d.

However, it does not end here. To find the derivative for functions that are products of two functions or are the result of one function being divided by another, we have the 'Product' and 'Quotient' rule. Also, one of the most important parts of differentiation is the chain rule. The chain rule suggests that while differentiation a function that has several functions in composite, if $y = g(u), where\ u = f(x)\ then, \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$; the derivative of the outer function multiplied by the derivative of the inner function. The three rules are shown below.

Chain rule	$y = g(u)$, where $u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
Product rule	$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

<u>Product Rule</u>

Example:

Q. Find the derivative of :

$$f(x) = x^2 sin\ x$$

Let,

$$u = x^2 \quad ; \quad v = sin\ x$$
$$\frac{du}{dx} = 2x \quad ; \quad \frac{dv}{dx} = cos\ x$$

Then,

$$f'(x) = (x^2 \times cos\ x) + (2x \times sin\ x) = x^2 cos\ x + 2x sin\ x$$

<u>Quotient Rule</u>

Example:

Q. Use the quotient rule to find $\frac{dy}{dx}$if

$$y = \frac{1 + 3x}{2 - x}$$

Let,

$$u = 1 + 3x \quad ; \quad v = 2 - x$$
$$\frac{du}{dx} = 3 \quad ; \quad \frac{dv}{dx} = -1$$
$$f'(x) = \frac{(2 - x) \times 3 - (1 + 3x) \times -1}{(2 - x)^2} = \frac{6x + 7}{(2 - x)^2}$$

<u>Chain Rule</u>

Example:

Q. Differentiate with respect to x:

$$f(x) = sin(x^2)$$

To solve such questions we need to make use of the chain rule. We first take the derivative of the outer function, which in this case in $sin\ x$, and then multiply it by the derivative of the inner function x^2.

$$f'(x) = cos(x^2)(2x) = 2x\ cos\ (x^2)$$

Exercise:

Q. Find the following:-

1. $\frac{d}{dx}(\frac{4x}{x-5})$
2. $\frac{d}{dx}(-\frac{x^2}{\sqrt{x^2+3}})$
3. $\frac{sin\ x}{x}$
4. $x\ tan\ x$
5. $log(cos\ 2x)$
6. $cos(cos\ x)$
7. $x\sqrt{cosec\ x}$
8. The point P has y-coordinate $\frac{\pi}{6}$, and lies on the graph of $f(x) = arctan\ (\frac{x}{3})$.

Find the coordinates of P

Find the gradient of tangent to $y = f(x)$ at P.

9. $x^2 cosec\ x$
10. $ln\ (x(x^2 + 1))$

<u>Second and Higher Order Derivatives</u>

Taking the second order derivative means differentiating the first derivative, or differentiating the function twice. If we were to find the n^{th} derivative, it would mean differentiating the function $n$times.

Exercise:

Q. If $y = 2e^{3x} + 5e^{4x}$,show that $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$.

Q. If $y = sin(2x + 3)$,show that $\frac{d^2y}{dx^2} + 4y = 0$.

Q. If $y = sin\ x$,show that $\frac{d^4y}{dx^4} = y$.

Q. If $y = 2sin\ x + 3cos\ x$, show that $y'' + y = 0$.

Q. Let $f(x) = e^{-x}(x + 2)$ .Find:

I. $f'(x)$

II. $f''(x)$

III. $f'''(x)$

IV. $f^{(4)}(x)$

V. Hence form a general formula for $f^{(n)}(x), n \in Z^+$.

<u>Implicit Differentiation</u>

The process of finding the derivative of a dependent variable in an implicit function by differentiating each term separately, by expressing the derivative of the dependent variable as a symbol, and by solving the resulting expression for the symbol.

Example:

Q. Find $\frac{dy}{dx}$if

a) $x^2 + y^2 = 25$
b) $x + \cos y = 1$

To solve this we need to carry out implicit differentiation in the following manner:-

a) $2x + 2y\frac{dy}{dx} = 0$

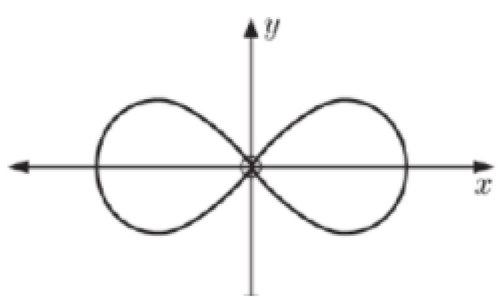
Once we different each of the terms with respect to 'x', we must rewrite it to make $\frac{dy}{dx}$the subject of the formula. Therefore, for a) $\frac{dy}{dx} = -\frac{x}{y}$.

b) $1 - \sin y(\frac{dy}{dx}) = 0$; $\frac{dy}{dx} = \frac{1}{\sin y}$

Exercise:

Q. Find $\frac{dy}{dx}$ if

1. $y + x\sec y = \arccos x$
2. $\ln(xy) = e^y$
3. $\sin y + xe^y = 2y$
4. The graph shows the relation $(x^2 + y^2)^2 = x^2 - y^2$.
   a) Find the axes intercepts.
   b) Find the coordinates of all points on the graph at which the tangent is horizontal.



Q. An ellipse has the equation $x^2 + 3y^2 = 48$. Find the coordinates of the points on the ellipse such that the normal at those points passes through (0,2).

<u>Properties of Curves</u>

1. Tangents and Normals

As mentioned before one of the key properties of differentiation is to find the equation of a tangent to a curve. The normal at that point is simply perpendicular to the tangent, and can be calculated since the slope of the normal is the negative reciprocal of the slope of the tangent.

Example:

Q. Find the equation of the tangent to:

a) $y = \frac{1}{\sqrt{3-2x}}$at $x = -3$

To do this we must differentiate the equation with respect to 'x'.

$$y' = \frac{-1}{2}(3-2x)^{-\frac{3}{2}} \times -2 = (3-2x)^{-1.5}$$

After this has been done, we must substitute -3 with 'x' to find the slope of tangent at x = -3.

$$y' = (3-2(-3))^{-1.5} = \frac{1}{27}$$

We know the slope of the tangent, however we need the equation in the form y = mx + c. We now substitute 3 in the function to obtain its corresponding 'y' value.

$y = \frac{1}{\sqrt{3-2(-3)}} = \frac{1}{3}$, therefore (-3,$\frac{1}{3}$) lies on the curve. Now, we can obtain the equation of the tangent by doing:-

$y - \frac{1}{3} = \frac{1}{27}(x+3)$

Example:

Q. Find the equation of another tangent to $y = 1 - 3x + 12x^2 - 8x^3$which is parallel to the tangent at (1,2).

In order to solve this, we first differentiate with respect to 'x'.

$$\frac{dy}{dx} = -3 + 24x - 24x^2$$

Slope of tangent = $-3 + 24(1) - 24(1) = -3$

$$-3 + 24x - 24x^2 = -3$$

$$24x(1-x) = 0$$

$$x = 1;\ x = 0$$

$x = 1$, was already given to us, however the second answer we obtained is the key to answering the question.

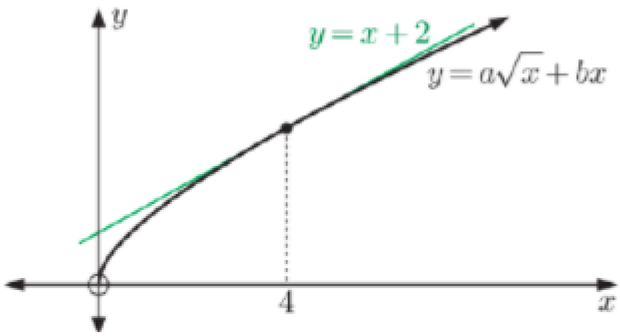
At x = 0, we need to find the corresponding 'y' value.

$y = 1 - 3(0) + 12(0)^2 - 8(0)^3 = 1$; (0,1)

Now we can determine the equation of the tangent since we have the slope and a point on the tangent. Thus, the equation would be

$$y - 1 = -3(x - 0)$$

$$y = -3x + 1$$



Find a and b.
To solve this question we first equate both the equations to get an equation in a and b.

$$x + 2 = a\sqrt{x} + bx$$

$$6 = 2a + 4b$$

$3 = a + 2b \rightarrow$ Eq 1

We know that the slope of the tangent to the curve at x = 4 is 1, since the coefficient of x in $y = x + 2$ is 1. Therefore, we can find another equation in terms of a and b.

$$y = a\sqrt{x} + bx$$

$$\frac{dy}{dx} = \frac{a}{2\sqrt{x}} + b$$

Slope of the tangent at x = 4 is 1.

$$1 = \frac{a}{2(2)} + b$$

$a + 4b = 4 \rightarrow$ Eq 2

Using these two simultaneous equations we can find the value of a and b.

$$a = 4 - 4b$$

$$a = 3 - 2b$$

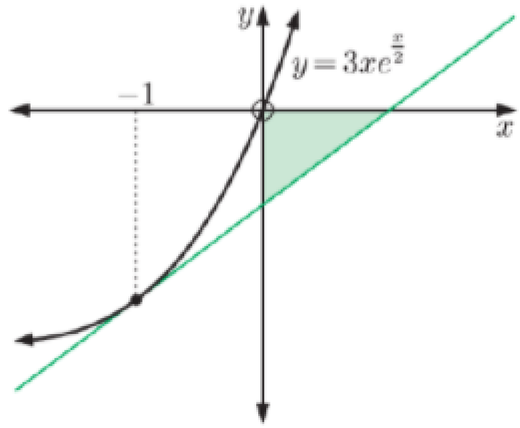
$$4 - 4b = 3 - 2b$$

$1 = 2b;\ b = \frac{1}{2}$ and therefore $a = 4 - 4(\frac{1}{2}) = 2; a = 2$

Exercise:

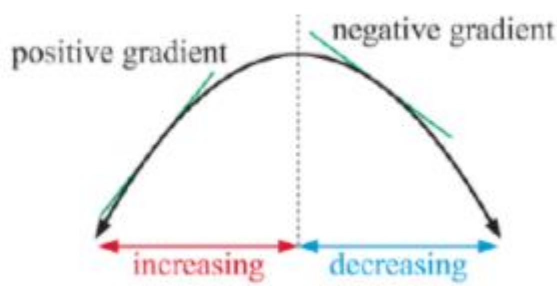
Q. Consider the curve $y = a\sqrt{1 - bx}$ where a and b are constants. The tangent to this curve at the point where $x = -1$ is $3x + y = 5$. Find the values of a and b.

Q. Find the exact area of the shaded triangle.



2. Increasing and Decreasing Functions

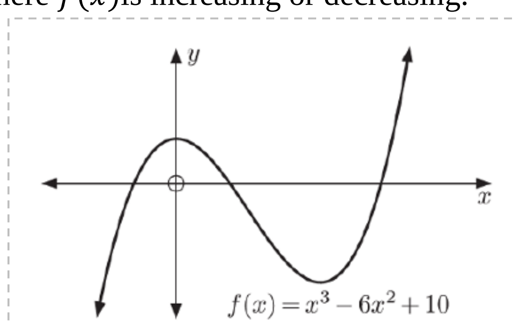
If $f(x)$is increasing then $f'(x) \geq 0$and a function $f(x)$is decreasing when $f'(x) \leq 0$.



Example:

Q. The graph of $f(x) = x^3 - 6x^2 + 10$.

a. Find $f'(x)$, and draw its sign diagram.
b. Find the intervals where $f(x)$is increasing or decreasing.



<u>**Note**</u>:- A sign diagram is very useful in determining the behaviour of a function.

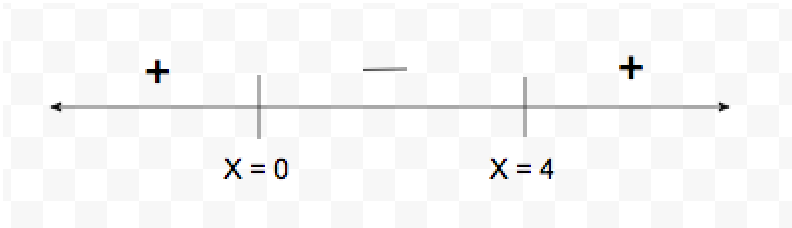
a. First we find the derivative of the function which is:-

$$f'(x) = 3x^2 - 12x$$

After we find the derivative we need to draw its sign diagram. To draw the sign diagram we must find the turning points. These are found by equating the derivative to 0.

$$3x^2 - 12x = 0$$
$$3x(x - 4) = 0$$
$$x = 0;\ x = 4$$

We substitute a point smaller than 0 in the derivative and a value of 'x' greater than 4 in the derivative, and a value between 0 and 4. If the value is positive we put a $+ve$ sign and if the value is negative we put $-ve$ sign. For example the sign diagram for the above derivative is the following:



b. To answer this we need to carefully observe the sign diagram. We know that if the derivative is positive, the function is increasing and if it is negative, it is decreasing. Therefore the function is increasing when x > 4 and x < 0, and the function is negative when 0 < x < 4.

Exercise:

Q. Find the intervals where $f(x)$ is increasing or decreasing:-

a) $f(x) = x^3 - 6x^2$
b) $f(x) = \frac{2}{\sqrt{x}}$
c) $f(x) = \sqrt{x}\, ln\, x$

Q. Consider $f(x) = \frac{4x}{x^2+1}$

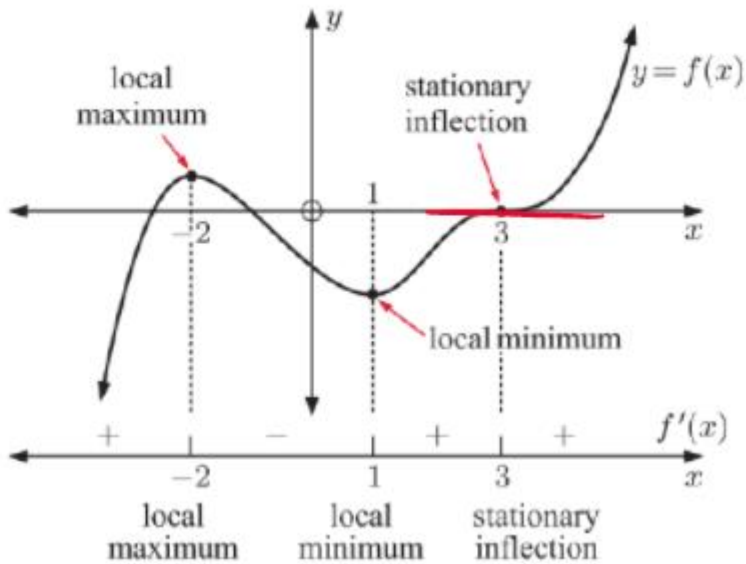
a) Show that $f'(x) = \frac{-4(x+1)(x-1)}{(x^2+1)^2}$and draw its sign diagram
b) Hence find intervals where $y = f(x)$is increasing or decreasing.

3. Stationary Points

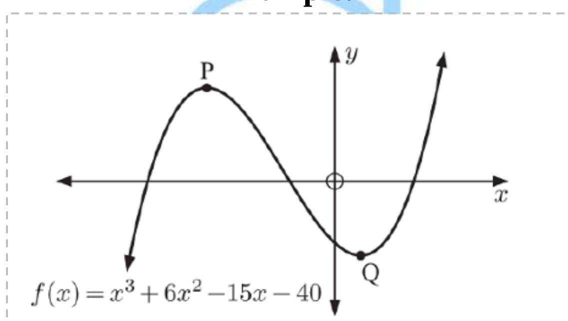
Although you did not know what stationary points were, in the previous sums we have been finding them all along. A stationary point of a function is a point where $f'(x) = 0$. It could be a **local maximum** or a **local minimum** or a **point of inflection.**

Note:- At a stationary point the tangent is horizontal (gradient = 0).

The diagram below explains what these three terms mean and the respective sign diagram for each of them.



Example:



Q. The graph of $f(x) = x^3 + 6x^2 - 15x - 40$is shown. P and Q are stationary points.

a) Classify points P and Q.
b) Find $f'(x)$
c) Find the coordinates P and Q
A. From our knowledge from functions we can classify point P as a maximum and point Q as a minimum. We can derive that $f'(x) = 3x^2 + 12x - 15$. Since stationary points are points at which the derivative is 0, to find the coordinates of points P and Q we equate the derivative to 0.

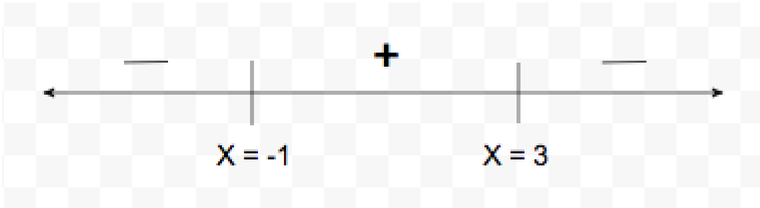
$f'(x) = 3x^2 + 12x - 15 = 0$. This gives us $x = -5$; $x = 1$. To find the corresponding y values we simply substitute the x values in the $f(x)$.

Q. Consider the function $g(x) = -2x^3 + 6x^2 + 18x - 7$.

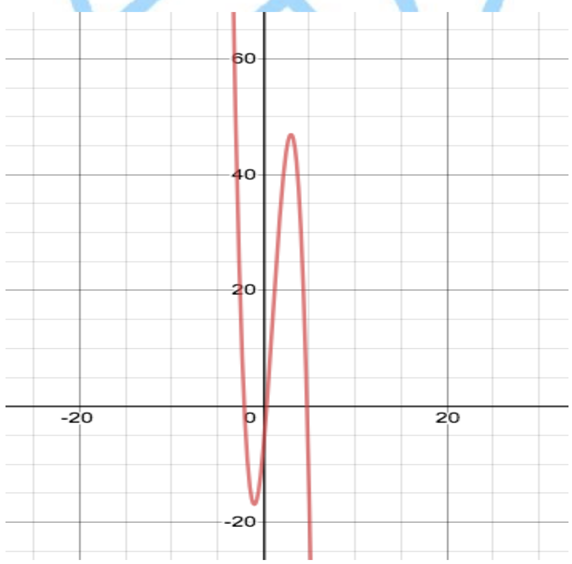
a) Find $g'(x)$,and draw its sign diagram.
b) Find intervals where the function is increasing and decreasing
c) Find and classify any stationary points
d) Sketch the graph $y = g(x)$showing the features you have found
e) Describe the behaviour of the function as $x \to \infty$and as $x \to -\infty$

We begin by differentiating the function: $g'(x) = -6x^2 + 12x + 18$. To draw the sign diagram we equate the derivative to 0.

$$-6x^2 + 12x + 18 = 0$$
$$x = 3;\ x = -1$$



Then we need to find the intervals where the function is increasing or decreasing. For this we need to see the sign diagram. Since the derivative is positive between -1 and 3, that is when the function is increasing. Since the derivative is negative when x > 3 and x < -1 it is decreasing during that time. Stationary points are when the derivative is equal to 0. Thus, the x-coordinate of the stationary points have already been determined: $x = 3;\ x = -1$. To find the corresponding 'y' values we have the substitute them in $y = g(x)$. To classify them we need to observe the sign diagram again. If at x = a the sign goes from negative to positive there is a local minimum at that point, and if at x = a the sign goes from positive to negative there is a local maximum at that point. However, if the sign is the same it is a point of inflection. Therefore, since the sign changes from negative to positive at x = - 1, there is a local minimum at that point, and because the sign changes from positive to negative at x = 3, there is a local maximum at that point.



Using the properties of the graph that we obtained we can easily sketch the above graph. Lastly, as $x \to \infty$we see that the graph is going in the downward direction where $y \to -\infty$.and as $x \to -\infty$in the second quadrant we see that the graph is moving in the upward direction where $y \to +\infty$.

Exercise:

Q. For each of the functions find and classify any stationary points. Sketch the function, showing all the important features.

a) $y = x^4 - 6x^2 + 8x - 3$

b) $y = x^2 e^x$

Q. The cubic polynomial $Q(x) = ax^3 + bx^2 + cx + d$ touches the line with the equation $y = 9x + 2$at the point (0,2) and has a stationary point at (-1,-7). Find $Q(x)$.

Q. Find the greatest and least value of

a) $x^3 - 12x - 2 \; for \; -3 \le x \le 5$

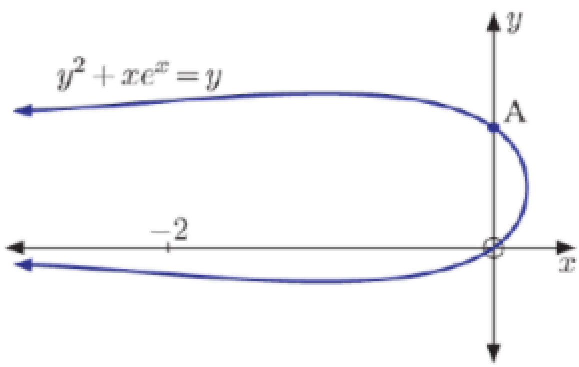
Q. Prove that $\frac{\ln x}{x} \le \frac{1}{e}$ $for\ all\ x > 0.$

Q. Consider the relation $y^2 + xe^x = y$.

a) Find $\frac{dy}{dx}$.

b) Find the equation of the normal to the equation at A.

c) Find the exact coordinates of the stationary points.



4. Shape

: we say that the curve is concave down

Note:- At this point $y'' > 0$.

: we say that the curve is concave up

Note:- At this point $y'' < 0$.

Example:

Q. For the given function determine the intervals on the function where it is:-

a) Increasing
b) Decreasing
c) Concave Up
d) Concave Down

$$f(x) = \frac{-1}{\sqrt{x}}$$

To solve this we need to find the derivative of the function.

$$f'(x) = \frac{1}{2}x^{-1.5}$$

Then, we equate the derivative to 0 to find the x-coordinate of the stationary points.

$\frac{1}{2}x^{-1.5} = 0$; However when we equate it to zero we realise that this equation does not have an answer, there is no solution to it, therefore there is no stationary point.

The domain of the function is $x > 0$ and for all positive values of x since $f'(x) > 0$, the function is always increasing, and therefore never decreasing.

Then to answer about the shape of the function we find the second derivative of the function or $f''(x)$. In this example -

$f''(x) = \frac{-3}{4x^2\sqrt{x}}$and for any positive value of x, $f''(x)$< 0. This means that the function is concave down.

Exercise:

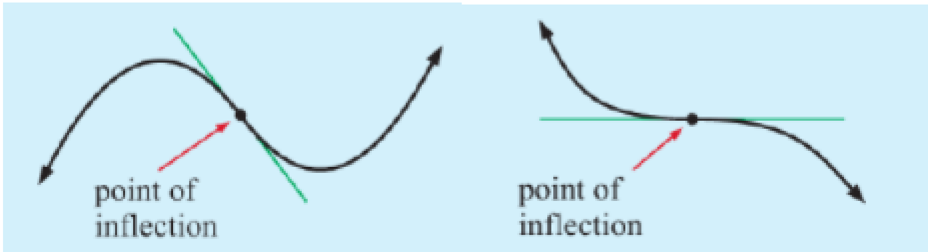
Q. Consider $f(x) = ln(2x - 1) - 3$.

a) Find the x-intercept
b) Can f(0) be found ? State its significance
c) Find the domain of the function
d) Find the gradient of the tangent to the curve at x = 1
e) Find , and hence explain why the function is concave down for all x in the domain of f
f) Graph the function showing the features you have found

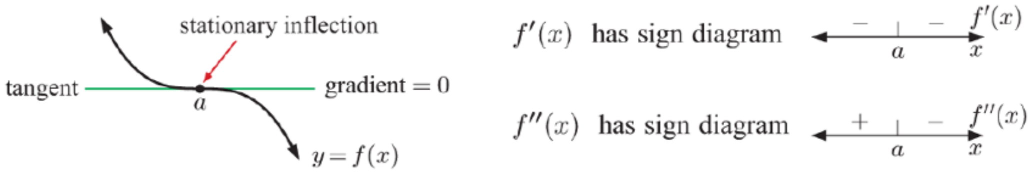
Q. Find intervals where the curve is concave up or concave down for $f(x) = \frac{x^2+x-3}{x+1}$.

5. Inflection Points

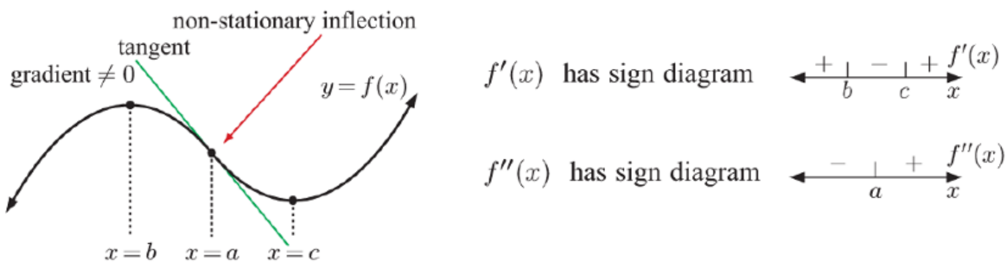
A point of inflection is a point at which the tangent to the curve crosses the curve. At a point of inflection $f''(x) = 0$.



If the tangent at a point of inflection is horizontal, then this is a **stationary inflection point**.



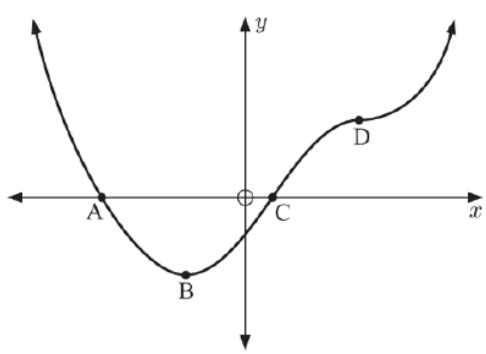
If the tangent at a point of inflection is *not* horizontal, then this is a **non-stationary inflection point**.



<u>Note:</u>- There is a point of inflection at x = a, if $f''(x) = 0$ and the **sign** of $f''(x)$ changes at x = a.
The point of inflection is a:

- Stationary inflection if $f'(x) = 0$
- Non-stationary inflection if $f'(x) \neq 0$

Example:



Q. Describe the inflection points of the graph.

A. C is a non-stationary point of inflection because the tangent to the curve at C would not be horizontal and therefore would not be equal to 0. D is a stationary point of inflection because the tangent to the curve at D would be horizontal and therefore equal to 0.

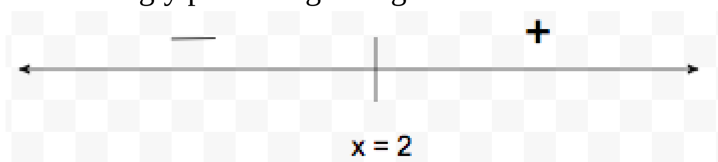
Q. Find and classify all points of inflection.

a) $y = x^3 - 6x^2 + 9x + 1$

First we find the derivative, followed by the second derivative.

$$y' = 3x^2 - 12x + 9$$
$$y'' = 6x - 12$$

Now, we equate the second derivative to 0 to obtain x = 2. Then we draw the sign diagram for the second derivative. For this we need to plug in one value greater than 2 and less than 2, in the second derivative and accordingly put the signs to get:



Therefore, there is a point of inflection at x = 2, since at this point the double derivative is equal to 0 **and** there is a sign change in the sign diagram for the double derivative.

To figure out whether it is a stationary or non-stationary inflection point we substitute 2 in the first derivative. $y' = 3x^2 - 12x + 9$

$y' = 3(2)^2 - 12(2) + 9 = -3$. Since $-3 < 0$, there is a non-stationary point of inflection at x = 2.

Exercise:

Q. For $f(x) = x^4 - 4x^2 + 3$

a) Find and classify all the turning points
b) Find and classify all points of inflection
c) Find where the function is increasing or decreasing
d) Find intervals where the function is concave up or down
e) Sketch the function showing the features you have found

Q. Consider the function $y = e^x - 3e^{-x}$

a) Determine the axes intercepts
b) Prove that the function is increasing for all 'x'.
c) Show that $\frac{d^2y}{dx^2} = y$.
d) Use the GDC to graph the function and its features you have found

Q. Consider the surge function $f(t) = Ate^{-bt}, t \geq 0$,where A and b are positive constants. Prove that the function has:

a) A local max at t = 1/b
b) A point of inflection at t = 2/b
c) Sketch the function showing the features you have proved

<u>Differentiability and Continuity</u>

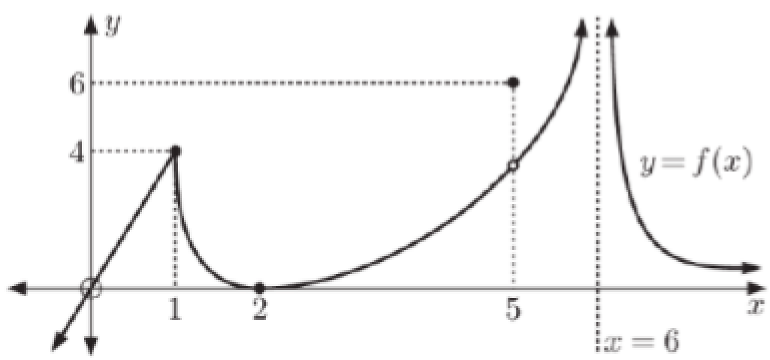
Continuity:
At a point:A function f(x) is said to be continuous at a point x=p, if $\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x)$,that is, the function is approaching the same y-value from both the left hand and right hand side of p.
During an interval: The function g(x) is said to be continuous during an interval (c,d) given that c<d only if g(x) is continuous at all points during the interval.
Standard examples of continuous functions:

1. All polynomial functions
2. Cosine and sine functions

Essential and removable discontinuities:
Functions have 3 types of discontinuities:

1. Hole
2. Jump
3. Break

Of these 3, a 'break' and a 'jump' is essential while a 'hole' is removable.



In this graph, there is a hole (aka a removable discontinuity) at x=5 and an essential discontinuity in the form of a break in the graph at x=6. We can remove the hole at x=5, by defining the value of f(x) as 4 when x=5 using a piecewise function.

Differentiability:
A function f(x) is said to be differentiable at a point x=a if

$$Right\ hand\ derivative = Left\ Hand\ Derivative$$

$Right\ hand\ derivative = \lim_{h \to 0} \frac{f(a^+ + h) - f(a^+)}{h}$ (where a^+ indicates a is being approached from the right hand side)

$Left\ hand\ derivative = \lim_{h \to 0} \frac{f(a^- + h) - f(a^-)}{h}$(where a^- indicates a is being approached from the left hand side)

Note:- All differentiable functions are continuous, but not all continuous functions are differentiable.

The overall function is differentiable if a derivative can be found on all points on the graph of the function.

Functions with cusps, sharp edges, or breaks are not differentiable. Examples: $y = |x|\ at\ x = 0$.

Note- This is merely an important concept. You will not be tested on this in an IB exam.

Limits and L'Hopital's Rule

When we discussed vertical asymptotes in Functions, we saw that the x-values of the function kept getting closer and closer to the vertical asymptote value (let it be noted by b), but were never quite equal to it. As the x-values keep approaching b, the function values also approach certain values. Limits help us determine what these values are.

We say that the limit of f(x) is L as x approaches b. This can be written in the form given below.

$$\lim_{x \to b} f(x) = L$$

Given that f(x) is as close to L as possible as we want x sufficiently close to b, from both sides, without actually letting x be b.

Example:

Find the limit: $\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 - 2x}$

Here, if we substitute x=2 in the expression, we get $\frac{0}{0}$.There are 2 ways we can solve this. Firstly, we can graph the function and observe the behavior of the graph as the x-value approaches 2. Secondly, we can use L'Hopital's Rule.

Given that f(x) and g(x) are differentiable and g'(x) is not equal to zero on an interval that contains the point x=a. If $\lim_{x \to c} f(x) = 0$and $\lim_{x \to c} g(x) = 0$or, if as f(x)→ $\pm\infty$and g(x)→ $\pm\infty$, then $\lim_{x \to d} \frac{f(x)}{g(x)} = \lim_{x \to d} \frac{f'(x)}{g'(x)}$, given that the limit on the right exists.

Thus, we differentiate the numerator and denominator of the above expression separately and then substitute the value of 2.

Note- A limit does not exist if the limit is ∞or if the value the function approaches as it approaches b from the left hand side is not equal to the value the function approaches as it approaches b from the right hand side.

Sample Exercise 2:

Find the limit: $\lim_{x \to 0} \frac{x^2}{1-\cos(x)}$. Here, if we substitute zero in the expression, we get $\frac{0}{0}$. Hence, we apply L'Hopital's Rule. However, even after applying the rule, we get $\lim_{x \to 0} \frac{2x}{\sin(x)}$ which still gives us $\frac{0}{0}$. Thus, we have to apply L'Hopital's Rule again to solve this.

Exercise:

1. $\lim_{x \to 0} \frac{x^2+2x}{1-\cos(x)}$
2. $\lim_{x \to 0} \frac{x+\sin x}{x-\sin x}$
3. $\lim_{x \to 0} \frac{a^x-b^x}{\sin x}$
4. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$
5. $\lim_{x \to 0^+} \frac{\ln(\cos(5x))}{\ln(\cos(3x))}$

Rates of Change

One of the many applications of differentiation includes 'Rates Of Change'. Here we examine how one variable changes with respect to another. Measuring the rate of something means how fast or slow it changes with respect to time.

Example:

Q. The future profit of a company is modelled as $P(x) = 2x^2 - 12x + 118$ thousand dollars, where x is the time in years from now.

a) Find $\frac{dP}{dx}$ at state its units
b) Find $\frac{dP}{dx}$ at x = 8 and state its meaning.

We have been given an equation that models profit of a company. The expression $\frac{dP}{dx}$ tells us how the profit of the company changes with respect to time. To find $\frac{dP}{dx}$ we simply differentiate P with respect to x.

$$\frac{dP}{dx} = 4x - 12$$

To find $\frac{dP}{dx}$ at x = 8, we substitute 8 in the derivative.

$4(8) - 12 = 20$; This means that the rate at which profit will be made in the 8th year will be 20,000 dollars per year.

Example:

Q. The temperature of a liquid after being placed in a refrigerator is given by $T = 5 + 95e^{-kt}$, where t is the time in minutes.

a) Show that $\frac{dT}{dt} = c(T - 5)$

This is an example of how the temperature of an object is changing with time, or one can say it shows the rate of change of T with t. First we differentiate the equation.

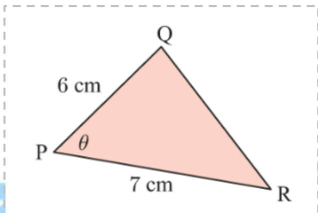
$$\frac{dT}{dt} = 0 + 95e^{-kt}(-k)$$

We can rewrite $T = 5 + 95e^{-kt}$ as $T - 5 = 95e^{-kt}$

Therefore, $\frac{dT}{dt} = (-k)(T - 5)$,where c = -k.

Example:

Q. Find exactly the rate of change in the area of triangle PQR as θ changes, at the time when $\theta = 45^{\circ}$.



In this example, we have not been given any formula or equation. However, we know that the area of a triangle is $A = \frac{1}{2}ab \cdot sin\theta$. We can substitute the two sides that we have been given. Thus, $A = 3(7)\ sinC = 21sin\theta$. Now we can differentiate A with respect to t. $\frac{dA}{dt} = \boldsymbol{21}cos\theta \cdot \frac{d\theta}{dt}$**;** Now we simply substitute 45 in place of θ to get $\frac{dA}{dt} = \frac{21}{\sqrt{2}} \cdot \frac{d\theta}{dt}$, and that is how we can find the rate of change of the Area with respect to the time as the angle changes.

<u>Optimisation</u>

A very useful application of calculus is optimisation. It involves maximizing or minimizing a variable to make efficient or best use of it. For example, a company would aim to maximize profit and minimize costs and therefore can model equations and find ways to accomplish their goals. Let us solve a few examples together.

Example:

Q. The total cost of producing x chocolates per day is $\frac{x^2}{4} + 8x + 20$dollars and for this production level each chocolate may be sold for $(23 - \frac{1}{2}x)$dollars. How many chocolates should be produced per day to maximise the total profit?

In this example we need to optimize the production level to find the best number of chocolates that should be produced per day for profits to be maximum. First of all, we need to frame an equation that gives us the profit. Let the profit $P(x)$.

Since profit = revenue - cost

$P(x) = x(23 - \frac{1}{2}x) - \frac{x^2}{4} + 8x + 20$. To maximize/minimize something we need to equate the derivative to 0. So by differentiating we get $P'(x) = 31 - \frac{3x}{2}$. Now we must equate this to zero. $31 = \frac{3x}{2}$; $3x = 62; x = 20.67$. We can check if this is actually a maximum by equating the double derivative to 0 and if it is a negative value it means that at that particular point there is a maximum.

Example:

Q. A cone has radius r cm and slant height s cm. Find the ratio s:r which maximises the volume of a cone.

$h = \sqrt{s^2 - r^2}$. We know that the volume of a cone is given by:-

$V = \frac{1}{3}\pi r^2 h$; We can now replace ‘h’ with $\sqrt{s^2 - r^2}$. Therefore, $V = \frac{1}{3}\pi r^2 \sqrt{s^2 - r^2}$.

Using the product rule,

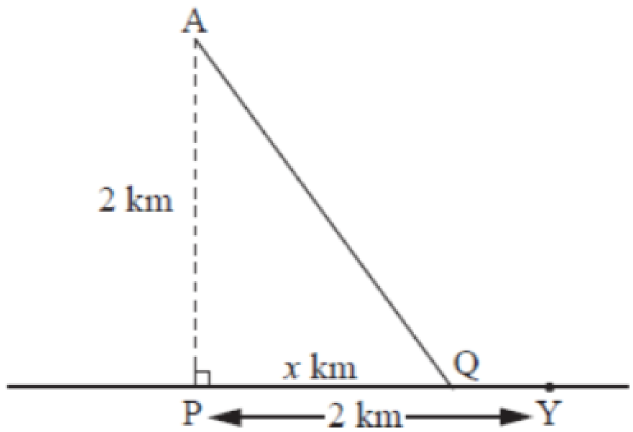
$\frac{dV}{dr} = \frac{2}{3}\pi r\sqrt{s^2 - r^2} - \frac{1}{3}\frac{\pi r^3}{\sqrt{s^2 - r^2}}$. Now we need to maximise the volume and therefore we must equate the derivative of the volume with respect to the radius.

$\frac{2(s^2 - r^2) - r^2}{\sqrt{s^2 - r^2}} = 0$; $2(s^2 - r^2) - r^2 = 0$; $2s^2 = 3r^2$; $\frac{s}{r} = \sqrt{\frac{3}{2}}$. Thus the ratio s: r is $\sqrt{3}$: $\sqrt{2}$.

Note:- The main idea in optimisation is to get an equation of the variable you want to maximise or minimise and find its derivative. Once you have done this, you need to equate it to 0 and you are good to go!

IB Questions:

Q. André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers 1 km in $5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

(a) If PQ = x km, $0 \leqslant x \leqslant 2$, find an expression for the time T minutes taken by André to reach point Y.

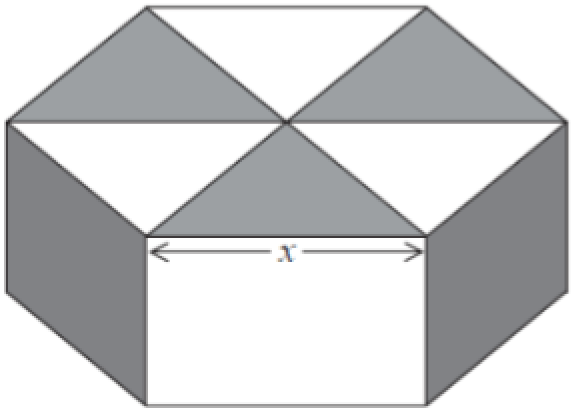
(b) Show that $\frac{dT}{dx} = \frac{5\sqrt{5x}}{\sqrt{x^2+4}} - 5$.

(c) (i) Solve $\frac{dT}{dx} = 0$

(ii) Use the value of x found in part (c) (i) to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\frac{d^2T}{dx^2} = \frac{20\sqrt{5}}{(x^2+5)^{1.5}}$ and hence show that the time found in part (c) (ii) is a minimum.

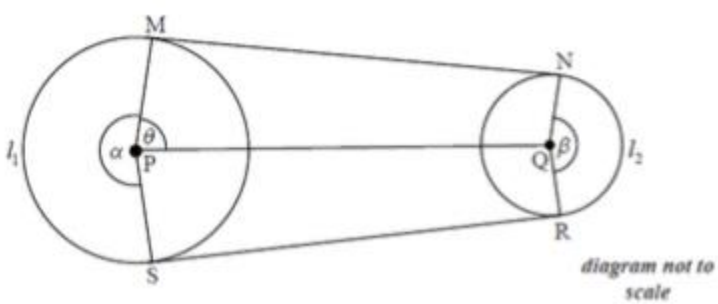
Q. A packaging company makes boxes for chocolates. An example of a box is shown below. This box is closed and the top and bottom of the box are identical regular hexagons of side x cm.



(a) Show that the area of each hexagon is $\frac{3\sqrt{3}x^2}{2}cm^2$.

(b) Given that the volume of the box is $90cm^3$, show that when $\sqrt[3]{20}$ the total surface area of the box is a minimum, justifying that this value gives a minimum.

Q. Two non-intersecting circles C1 , containing points M and S , and C2 , containing points N and R, have centres P and Q where PQ = 50 . The line segments [MN] and [SR] are common tangents to the circles. The size of the reflex angle MPS is α, the size of the obtuse angle NQR is β , and the size of the angle MPQ is θ . The arc length MS is $l1$ and the arc length NR is $l2$. This information is represented in the diagram below.



The radius of C1 is x, where x ≥10 and the radius of C2 is 10.

a) Explain why $x < 40$
b) Show that $cos\theta = \frac{x-10}{50}$
c) (i) Find an expression for MN in terms of x (ii) Find the value of x that maximises MN.

d) Find an expression in terms of x for $\alpha$and β.
e) Find an expression, $b(x)$, for the length of the perimeter in terms of x
f) Find the maximum value of the length of the perimeter.
g) Find the value of x that gives a perimeter length of 200.

<u>Related Rates of Change</u>

Essentially, in related rates, our two variables (let's call them x and y) are changing with time. Thus, the problems in these types of questions is, in simple terms, determining the change in x with respect to time when we know the change in y with respect to time and a relation between x and y is given. Let us solve some sample questions together.

Example:

1. p and q are variables related by the equation $pq^3 = 30$.
    a. Differentiate this equation with respect to time t.
    b. When p=3, q is increasing at a rate of 1 unit per second. Find the rate of change of p.

So, we differentiate this equation using the concepts of implicit differentiation and product rule. This becomes: $p^3 \frac{dq}{dt} + 3p^2 q. \frac{dp}{dt} = 0$. Then, we need to find the rate of change of q. We know that $\frac{dq}{dt} = 1$. We can easily find q as p=3, and it is given that $pq^3 = 30$. Thus, we substitute the value of q in the equation we found and get the rate of change of p.
Note- The rate of change of p will be negative and that means that p is decreasing as q is increasing, something that we can easily deduce if we rewrite the original relation.

Example:

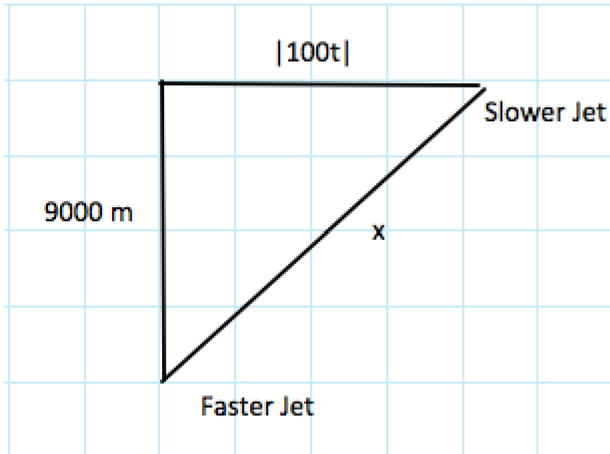
2. The length of a rectangle is decreasing at 2 cm per minute. However, the area remains constant at 300 cm^2. Taking c as the length and d as the width of the triangle:
    a. Determine the relation between c and d
    b. Differentiate the relation obtained in a with respect to time
    c. At what rate is the width of a rectangle increasing at the moment when
        i. c = 8
        ii. The rectangle is a square

Firstly, we know that the area of a rectangle is length into width. Thus, $cd = 300$.We then use implicit differentiation and the product rule to get the differentiated equation with respect to the time. Once we have achieved that, it is simple substitution.

Example:

3. Two planes fly on parallel courses that are 9 km apart. Their air speeds are 300 ms^{-1} and 400 ms^{-1}. How quickly is the distance between them changing at the moment when the faster jet is 3 km behind the slower one?

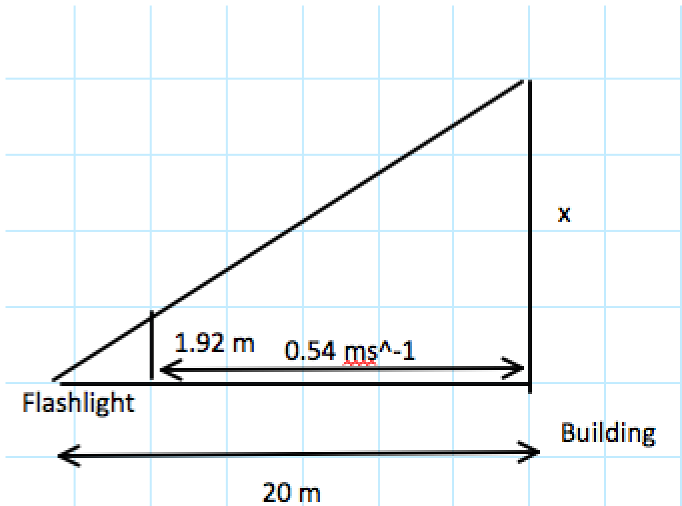
This is a slightly more complex question and we need to draw a diagram to understand the situation better. Firstly, we must understand that the relative speed of the two airplanes is 400-300=100 ms^{-1}. Thus, the distance that this covers in time t is given by d=100t.



Thus, we can deduce that $x^2 = (100t)^2 + 9000^2$. Once we have rewritten this by applying a square root on both sides of the equation, we must differentiate the equation in terms of t. We find the time by equating 100t to 9000 to get the time. Using this, we can easily find the rate of change of the distance between the two planes.

Example:

4. A flashlight level with the ground is located 20 m from the foot of a building. The flashlight shines in the building's direction. A 1.92 m tall person walks from the flashlight in the building's direction at .54 ms^{-1}. At what rate is the person's shadow shortening at the point when they are
   a. 16 m
   b. 9 m from the building?



$$\frac{x}{1.92} = \frac{20}{20 - 0.54t}$$

We then rewrite the equation in terms of x and differentiate with respect to t. For the two parts, we find the time taken by first finding the distance travelled by subtracting distance from the building from the total distance. Then, we use $t = \frac{d}{v}$ where we just found d and v is 0.54 ms^{-1}.

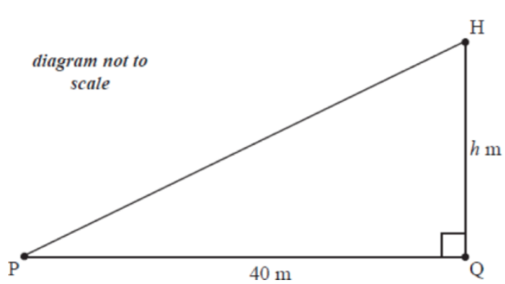
IB Questions:

1. *[7 marks]*

A ladder of length 10 m on horizontal ground rests against a vertical wall. The bottom of the ladder is moved away from the wall at a constant speed of . Calculate the speed of descent of the top of the ladder when the bottom of the ladder is 4 m away from the wall.

2. *[7 marks]*

A helicopter H is moving vertically upwards with a speed of 10 ms^{-1} . The helicopter is m directly above the point Q which is situated on level ground. The helicopter is observed from the point P which is also at ground level and PQ =40 m. This information is represented in the diagram below.



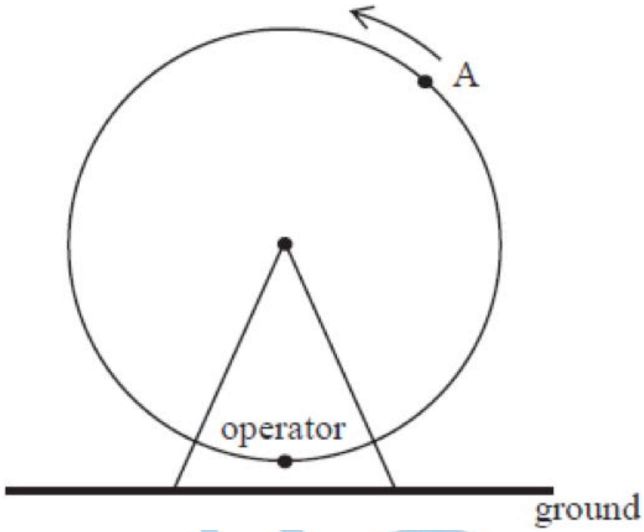
When , h = 30 m

(a) show that the rate of change of $H\widehat{PQ}$ is 0.16 radians per second;

(b) find the rate of change of PH.

3. *[10 marks]*

Below is a sketch of a Ferris wheel, an amusement park device carrying passengers around the rim of the wheel.

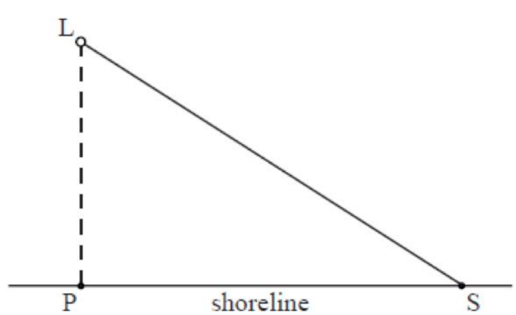


(a) The circular Ferris wheel has a radius of 10 metres and is revolving at a rate of 3 radians per minute. Determine how fast a passenger on the wheel is going vertically upwards when the passenger is at point A, 6 metres higher than the centre of the wheel, and is rising.

(b) The operator of the Ferris wheel stands directly below the centre such that the bottom of the Ferris wheel is level with his eyeline. As he watches the passenger his line of sight makes an angle with the horizontal. Find the rate of change of at point A.

4. *[8 marks]*

A lighthouse L is located offshore, 500 metres from the nearest point P on a long straight shoreline. The narrow beam of light from the lighthouse rotates at a constant rate of radians per minute, producing an illuminated spot S that moves along the shoreline. You may assume that the height of the lighthouse can be ignored and that the beam of light lies in the horizontal plane defined by sea level.



When S is 2000 metres from P,

(a) show that the speed of S, correct to three significant figures, is metres per minute;

(b) find the acceleration of S.

5. *[6 marks]*

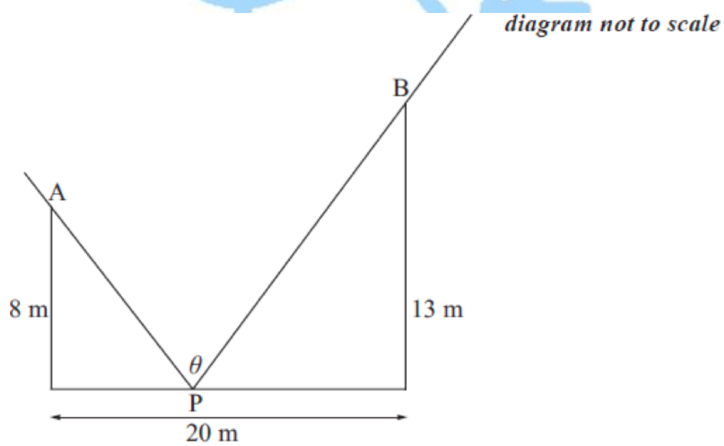
A stalactite has the shape of a circular cone. Its height is 200 mm and is increasing at a rate of 3 mm per century. Its base radius is 40 mm and is decreasing at a rate of 0.5 mm per century. Determine if its volume is increasing or decreasing, and the rate at which the volume is changing.

6. *[6 marks]*

Paint is poured into a tray where it forms a circular pool with a uniform thickness of 0.5 cm. If the paint is poured at a constant rate of $4 \text{ cm}^3\text{s}^{-1}$, find the rate of increase of the radius of the circle when the radius is 20 cm.

7a. *[2 marks]*

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle θ where $\theta = \text{A}\hat{\text{P}}\text{B}$, as shown in the diagram.



Find an expression for in terms of *x*, where *x* is the distance of P from the base of the wall of height 8 m.

7b. *[2 marks]*

(i) Calculate the value of θ when $x = 0$.

(ii) Calculate the value of θ when $x = 20$.

7c. *[2 marks]*

Sketch the graph of θ , for $0 \leq x \leq 20$

7**d.** *[6 marks]*

Show that :

$$\frac{d\theta}{dx} = \frac{5(744-64x-x^2)}{(x^2+64)(x^2-40x+569)}$$

7e. *[3 marks]*

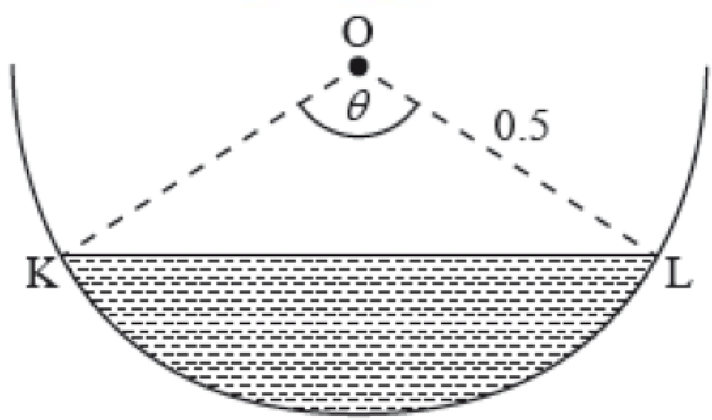
Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures.

7f. *[4 marks]*

The point P moves across the street with speed 0.5 m/s. Determine the rate of change of $\theta$with respect to time when P is at the midpoint of the street.

8a. *[3 marks]*

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is θ radians.



Find an expression for the volume of water in the trough in terms of θ.

8b. *[4 marks]*

The volume of water is increasing at a constant rate of $0.0008\ \mathrm{m^3s^{-1}}$.

Calculate $\frac{d\theta}{dt}$ when $\theta = \frac{\pi}{3}$.

PART B

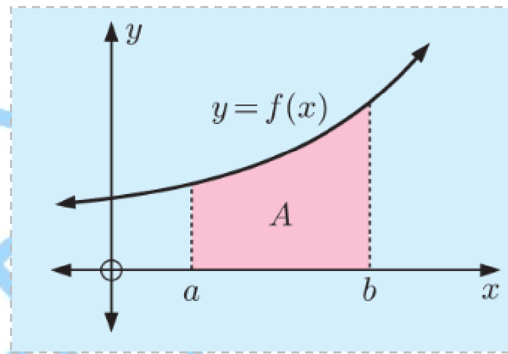
Integration:

Integration is the complete opposite of differentiation. If differentiating a function helps you find its slope at a particular point, integration helps you approximate the area and volume under a curve. Integration is also known as antidifferentiation.

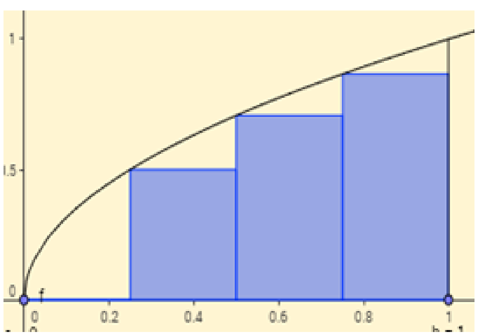
<u>Introduction to Integration</u>

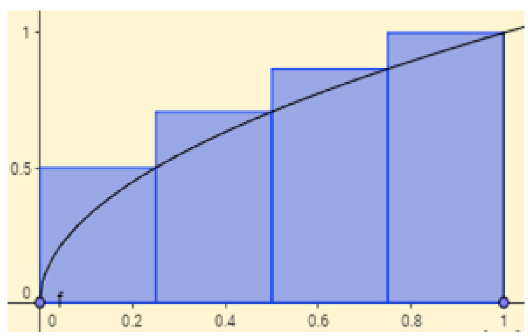
- Riemann Integral

If $f(x) \geq 0$ for all $a \leq x \leq b$, then
$\int_a^b f(x)\,dx$ is defined as the shaded area A.
This is known as the Riemann integral.
We would say “the integral of f(x) from a to b with respect to x”.



To approximate the area under the curve we can use several methods. One of the methods is calculating the area of lower and upper rectangles.





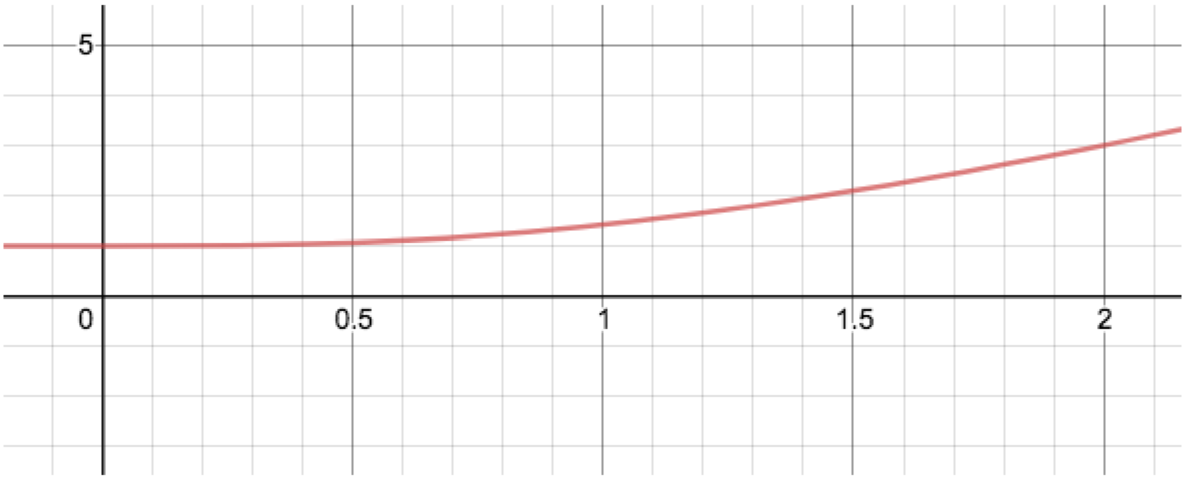
The first image shows lower rectangles and how finding the sum of their areas can give an approximate value for the area under the graph. The similar is shown in the second image with upper rectangles.

Example:

Consider the region enclosed by $y = \sqrt{1 + x^3}$ and the x-axis for $0 \leq x \leq 2$.

a. Write an expression for the lower and upper rectangle sums using n subintervals.
b. Find the lower and upper rectangle sums for n=50.
c. Hence estimate $\int_0^2 \sqrt{1 + x^3}\,dx$.

The graph for the function is:-



If we take Δx as the width of each rectangle and n as the number of rectangles then we can say that:-

$\Delta x = \frac{b-a}{n}$; where the area needs to be calculated between x = a and x = b.

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Therefore for $n = 50$

$$\Delta x = 0.04$$

To make our lives easier, there is a formula that allows us to calculate upper and lower rectangle sums for increasing and decreasing functions.

For an increasing function:-

$A_L = \sum_{k=1}^{n} \quad f(a + (k-1)\Delta x) \cdot \Delta x$ (sum of lower rectangles),

$A_U = \sum_{k=1}^{n} \quad f(a + k\Delta x) \cdot \Delta x$(sum of upper rectangles),

For a decreasing function:-

$A_L = \sum_{k=1}^{n} \quad f(a + k\Delta x) \cdot \Delta x$(sum of lower rectangles),

$A_U = \sum_{k=1}^{n} \quad f(a + (k-1)\Delta x) \cdot \Delta x$ (sum of upper rectangles),

Since, the above function is increasing we use the formulas for an increasing function. Using the formulas and substituting appropriately we get:-

$$A_L = 3.2016$$
$$A_U = 3.2816$$

Note:- It is helpful to remember that for an increasing function:-

$$A_L \leq actual\ area \leq A_U,$$

And for a decreasing function:-

$$A_U \leq actual\ area \leq A_L$$

Techniques of Integration

1. Rules of Integration

Integral of x^n	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
Area between a curve $y = f(x)$ and the x-axis, where $f(x) > 0$	$A = \int_a^b y \, dx$

To find the integral of a function we use the formula provided above.

Example:

Q. Find

a) $\int \quad (x^2 + 3x - 2)\, dx$

Using the formula above:-

$$\int \quad (x^2 + 3x - 2)\, dx = \frac{x^3}{3} + \frac{3x^2}{2} - 2x + c$$

Note:- ‘c’ is known as the constant of integration that must be added after integrating any indefinite integral.

b) $\int \quad (2x^2 - 3x + 1)\, dx = \frac{2x^3}{3} - \frac{3x^2}{2} + x + c$

c) $\int \quad (x\sqrt{x} - 9)\, dx = \frac{2x^{2.5}}{5} - 9x + c$

However, not all functions are so simple. For different functions there are different integrals. Some of the standard integrals have been listed below.

Standard integrals	$\int a^x \, dx = \frac{1}{\ln a} a^x + C$ $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C$, $\lvert x \rvert < a$

Standard integrals

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int e^x\,dx = e^x + C$$

Below are the properties of integrals:-

(1) $\int_a^a f(x)\,dx = 0$

(2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(3) $\int_a^b k\,dx = k(b-a)$

where k is constant.

(4) $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

(5) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(6) If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$

(7) If $f(x) \leq 0$ on $[a,b]$ then $\int_a^b f(x)\,dx \leq 0$

(8) If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Exercise:

Q. Find

a) $\int (2e^x - 3x)\,dx$
b) $\int (3\sin x - 2)\,dx$
c) $\int \left(\frac{x(x-1)}{3} + \sec^2 x\right)dx$
d) $\int 4^x dx$
e) $\int \left(\frac{3}{x} - \frac{1}{x\ln 2}\right)$
f) $\int \left(\frac{4}{\sqrt{1-x^2}}\right)dx$

<u>Note:-</u> If in place of x there is a linear model the rules can be modified as the following:-

1. $\int (ax+b)^n dx = \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1} + c$, for $n \neq -1$ and $a \neq 0$.

2. $\int \quad cos(ax+b)\,dx = \frac{1}{a} \cdot sin(ax+b) + c$
3. $\int \quad -\,sin(ax+b)\,dx = \frac{1}{a} \cdot cos(ax+b) + c$
4. $\int \quad sec^2(ax+b)\,dx = \frac{1}{a} \cdot tan(ax+b) + c$
5. $\int \quad \frac{1}{ax+b}\,dx = \frac{1}{a} ln|ax+b| + c$for $a \neq 0$
6. $\int \quad e^{ax+b} \quad = \frac{1}{a} e^{ax+b} + c$

Exercise:

Q. Find

a) $\int \quad (2x+5)^3 dx$
b) $\int \quad \frac{4}{\sqrt{3-4x}}\,dx$
c) $\int \quad (2sin(3x) + 5cos(4x))\,dx$
d) $\int \quad 3^{2x-1} dx$
e) $\int \quad \frac{1}{(2x+5)ln\,3}\,dx$

<u>Partial Fractions and integration</u>

Earlier, we had studied how to write a fraction as the sum of its partial fractions. The main reason to write it in such a form is so that it can be integrated easily.

Example:

$\int \quad \frac{2x-8}{x^2-4}\,dx$; We cannot directly integrate this, nor is there any formula that allows us to do so. Therefore, we must rewrite it in such a way that we can individually integrate the fractions and add their integrals up to get the answer.

As studied previously, $\frac{2x-8}{x^2-4}$can be written as $\frac{3}{(x+2)} - \frac{1}{(x-2)}$. Now this form is familiar for us, and we can easily integrate it.

$$\int \quad \frac{3}{(x+2)} - \frac{1}{(x-2)}\,dx = 3ln(x+2) - ln(x-2) + c$$

This can be rewritten as '$ln\left|\frac{(x+2)^3}{(x-2)}\right| + c$.'

Exercise:

a) $\int \quad \frac{6x^2+x-19}{(x+3)(x-1)^2}\,dx$
b) $\int \quad \frac{2}{4x^2-1}\,dx$

c) $\int \frac{20}{2x^2-x-3}\,dx$

d) $\int \frac{x-9}{x^2-2x-3}\,dx$

2. <u>Substitution Method</u>

Let us take a simple example:

$$\int 3x^2(x^3+1)^4 dx \text{ using } u = x^3+1.$$

The derivative of u can be found and it is as follows:
$\frac{du}{dx} = 3x^2$ and we can rewrite this to $\frac{du}{3x^2} = dx$.
Let us try utilising u in the integral:
$\int 3x^2 \times dx \times u^4$. However, dx can be written as $\frac{du}{3x^2}$. Thus, if we substitute this back into the integral, $3x^2$ cancels out and we are left with $\int u^4 du$.We can easily integrate this to get $\frac{u^5}{5}$. To end the sum, we need to rewrite u in terms of x and since that is given to us, we can conclude that:$\int 3x^2(x^3+1)^4 dx$=$(x^3+1)^5+c$.
Let us try the same concept with the integral: $\int 2xe^{x^2} dx$.We substitute $u=x^2$ and follow the exact same methodology as in the previous sum. $\frac{du}{dx} = 2x$ and we rewrite this as $\frac{du}{2x} = dx$.
Rewriting the integral in terms of u: $\int e^u du$ as 2x cancels out. Thus, $\int 2xe^{x^2} dx = e^{x^2} + c$.

Exercise:

Integrate by substitution

1. $\frac{3x^2}{\sqrt{x^3+2}}$
2. $2x\sin(x^2-5)$
3. $\frac{6x^2}{(2x^3-1)^6}$
4. $(2x-1)e^{x-x^2}$
5. $\frac{1-x^3}{x^2-3x}$
6. $\frac{e^x+e^{-x}}{e^x-e^{-x}}$(Hint, try using the denominator)
7. $\frac{x^3}{1+x^2}$(Hint, try using the denominator and rewriting the equation to make x^2 in terms of u)
8. $\frac{(\ln x)^2}{x}$
9. $\sin x 2^{\cos x}$
10. $e^{\tan x}.(\sec x)^2$
11. $e^{\cot x}.(\mathrm{cosec}\, x)^2$
12. Tan 3x
13. $\frac{x^2\ln(x^3+5)}{x^3+5}$

14. $x\sqrt{x-6}$ (Hint, try using the term under the bracket and using a part of the hint of sum 7)
15. $3x^2\sqrt{x-2}$

Now, we will look at integrating using substitution for trigonometric functions.

1. If an integral contains $\sqrt{a^2-x^2}$,try substituting $x = asin\theta$
2. If an integral contains $\sqrt{x^2+a^2}$try substituting $x = atan\theta$
3. If an integral contains $\sqrt{x^2-a^2}$try substituting $x = asec\theta$

Example:

$\int \sqrt{9-x^2}$. Here, we use the substitution mentioned above of $x = asin\theta$and a=3. Then, we differentiate x with respect to θ for further substitution.

We get$\frac{dx}{d\theta} = 3cos\theta$

Eventually, using all our information, we substitute $3sin\theta$into the integral.

$$\int \sqrt{9-9(sin\theta)^2}.3cos\theta\ d\theta$$

When we simplify the root using $(sin\theta)^2 + (cos\theta)^2 = 1$and eventually get $\int 3(cos\theta)^2$. In trigonometry, while we were learning the double angle formula, we learned that $cos2\theta = 2(cos\theta)^2 - 1$. We can use this formula to simplify the integral and get $\int \frac{9}{2}(1 + cos2\theta)$ From here, we can use the standard rules of integration to get our solution in terms of θ. Then, we must refer back to our original substitution and the double angle formula again to get the final answer.

Exercise:

Integrate with respect to x:

1. $\frac{x^2}{x^2+4}$(hint, try adding $\pm 4$to the numerator)
2. $\frac{x^2}{4-x^2}$
3. $\frac{3lnx}{x(1+(lnx)^2)}$

IB Questions:

1. Let $f(x) = \sqrt{\frac{x}{1-x}}, 0 < x < 1$
   a. Show that $f'(x) = \frac{1}{2}x^{\frac{-1}{2}}(1-x)^{\frac{-3}{2}}$and deduce that f(x) is an increasing function
   b. Show that the curve y=f(x) has one point of inflection and find its coordinates
   c. Use the substitution $x = (sin\theta)^2$ to prove that $\int f(x)\ dx = arcsin\sqrt{x} - \sqrt{x-x^2} + c$
2. Using the substitution$t = tan\theta$, find the integral $\int \frac{1}{3(cos\theta)^2+(sin\theta)^2} d\theta$
3. Using the substitution $u = 1+\sqrt{x}$, find $\int \frac{\sqrt{x}}{1+\sqrt{x}} dx$

4. By using the substitution $x^2 = 2sec\theta$, prove that $\int \frac{1}{x\sqrt{x^4-4}} = \frac{1}{4}arccos(\frac{2}{x^2}) + c$
5. Use the substitution $u = x^{\frac{1}{2}}$to find $\int \frac{1}{x^{\frac{3}{2}}+x^{\frac{1}{2}}}dx$.

3. <u>Integration by parts</u>

Integration by parts	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ or $\int u\,dv = uv - \int v\,du$

Integration by parts is one of the techniques of performing integration. There are two types of integration by parts which will be explored in this topic.

Example:

Q. Find

I. $\int xsinx\,dx$

To do this kind of integration we need to use a special technique. We need to separate this expression into two parts:-

$\int (x) \cdot (sin\,x)\,dx$; The next step is to let one of these be 'u' and the other 'dv'. Now the interesting part is how does one decide which of them should be 'u' and which should be 'dv'. There is an acronym called "L I A T E" which helps you decide which of them should be 'u'.

L : Logarithmic function
I : Inverse function
A : Algebraic function
T : Trigonometric function
E : Exponential function

Whichever function appears first is equated to 'u'. For instance in the example above 'x' is an algebraic function and 'sin x' is a trigonometric function. Because A comes before T in the acronym, we let x = u, and solve the sum in the following manner:-

Let x = u　　　　　　　　dv = sin x
x' = 1　　　　　　　　　v = - cos x

Now, using the formula we solve it.

$$\int xsinx\,dx = -xcos\,x + \int cos\,x\,dx$$

$$\int x\,sinx\,dx = -x\,cos\,x + sin\,x + c$$

In this manner, various expressions can be integrated using this method.
Let us understand a few more examples.

Q. Find $\int$ ln x dx

We can rewrite this as $\int \quad (1)(\ln x)\, dx$

$let\ u = \ln x \qquad\qquad dv = 1$

$u' = \frac{1}{x} \qquad\qquad v = x$

Now, using the formula we get:

$$\int (\ln x)\, dx = x \ln x - \int 1\, dx = x \ln x - x + c$$

Another type of integration sum can be of the repetitive type. This is an important part of calculus that has various applications. An example has been shown below.

Q. Find $\int \quad e^{-2x} \cos 2x\, dx$

$Let\ u = \cos 2x \qquad\qquad dv = e^{-2x}$

$u' = -2\sin(2x) \qquad\qquad v = \frac{e^{-2x}}{-2}$

Using the formula we get:-

$$\int e^{-2x} \cos 2x = \frac{-1}{2} e^{-2x} \cos(2x) - \int \sin(2x) e^{-2x}\, dx$$

We cannot stop here, we need to further integrate $\int \quad \sin(2x) e^{-2x}\, dx$, so we perform integration by parts again on $\int \quad \sin(2x) e^{-2x}\, dx.$

To solve this we let

$u = \sin(2x)$ and $dv = e^{-2x}$

$u' = 2\cos(2x) \qquad v = \frac{e^{-2x}}{-2}$

Using the formula we get:-

$$\int \sin(2x) e^{-2x}\, dx = \frac{-1}{2} e^{-2x} \sin(2x) - \int e^{-2x} \cos 2x\, dx$$

Putting it all together we get:-

$$\int e^{-2x} \cos 2x = \frac{-1}{2} e^{-2x} \cos(2x) - \int \sin(2x) e^{-2x}\, dx$$

$\int \quad e^{-2x} \cos 2x = \frac{-1}{2} e^{-2x} \cos(2x) - (\frac{-1}{2} e^{-2x} \sin(2x) - \int \quad e^{-2x} \cos 2x\, dx)$

However if we look closely we realise that we have ended up with the exact expression that we want to integrate.Therefore, we can call this expression I.

$I = \int \quad e^{-2x} \cos 2x\, dx)$; If we do that we get:-

$I = \frac{-1}{2} e^{-2x} \cos(2x) - (\frac{-1}{2} e^{-2x} \sin(2x) - I$

$$2I = \frac{-1}{2} e^{-2x} \cos(2x) + \frac{1}{2} e^{-2x} \sin(2x)$$

$2I = \frac{1}{2} e^{-2x} (\sin(2x) - \cos(2x))$

$$I = \frac{1}{4} e^{-2x} (\sin(2x) - \cos(2x)) + c$$

IB Questions:

1. Find $\int arcsin\, x\, dx$
2. Find $\int x^2 ln\, x\, dx$
3. Using integration by parts find $\int xsinx\, dx$
4. Find$\int x(secx)^2\, dx$
5. Find $\int x^3 \sin 2x\, dx$
6. Consider the functions f, g defined for $x \in R$, given by $f(x) = e^{-x} sinx\ and\ g(x) = e^{-x} cosx$
    a. Find f'(x)
    b. Find g'(x)
    c. Find $\int e^{-x} sinx\, dx$

<u>Definite Integrals</u>

$$\int_a^b f(x)\, dx = F(b) - F(a)\ where\ F(x)\ is\ the\ integral\ of\ f(x).$$

Note- when using substitution in definite integrals, make sure to obtain the new values that you will be integrating with respect to.

Example:

$\int_1^2 2x(x^3 - 1)^4 dx$. We use the normal steps for substitution of taking u=$x^3 - 1$and differentiating u with respect to x. However, we also must find the value of u at x=1 and at x=2, to make the calculation easier.

- **Area under a curve**

Area of region enclosed by a curve and x-axis	$A = \int_a^b \lvert y \rvert dx$
Area of region enclosed by a curve and y-axis	$A = \int_a^b \lvert x \rvert dy$

Example:

When the graph of f(x) is positive and continuous;

$y = lnx$. Find the area of the region from x=1 to x=4.

$$Area = \int_a^b f(x)\, dx$$

$$Area = \int_1^4 ln\, x\, dx$$

$$Area = (xlnx - \int dx)_1^4 = 4ln4 - ln1 - (x)_1^4 = 4ln4 - 3$$

Example:

When the graph of f(x) is negative and continuous;

$$Area = -\int_a^b f(x)\, dx$$

The area of the region bounded by $f(x) = -\frac{9}{x}$, the line x=3 and x=k is 9ln2 units. Find k.

$$-\int_3^k f(x)dx = 9ln2$$

$$-\int_3^k -\frac{9}{x}dx = 9ln2$$

Example:

Area between the y-axis and the curve;

$y = \frac{1}{\sqrt{x}}$. Find the area of the region bounded by the y-axis and the curve from y=1 to y=3.

$$\sqrt{x} = \frac{1}{y}.$$

$$x = \frac{1}{y^2}$$

$$Area = \int_1^2 \frac{1}{y^2}\, dy$$

Example:

Area between 2 functions

In order to find the area enclosed between 2 functions, we must first find the points of intersection between the two functions and then we can get the area using the formula:

$$Area = \int_a^b f(x) - g(x)\, dx$$

Find the area enclosed by $y = -x^3 + 3x^2 + 6x + 8$and $y = 3x - 5$.
$-x^3 + 3x^2 + 6x + 8 = 3x - 5$.

IB Questions:

1. The function f is defined on the domain $[0, \frac{3\pi}{2}]$ by $f(x) = e^{-x}cosx$
    a. State the two zeros of f .
    b. Sketch the graph of f .
    c. The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B . Show that the ratio of the area of A to the area of B is $\frac{e^{\pi}(e^{\frac{\pi}{2}}+1)}{e^{\pi}+}$
2. The diagram below shows the two curves $y = \frac{1}{x}$and $y = \frac{k}{x}$where k>1

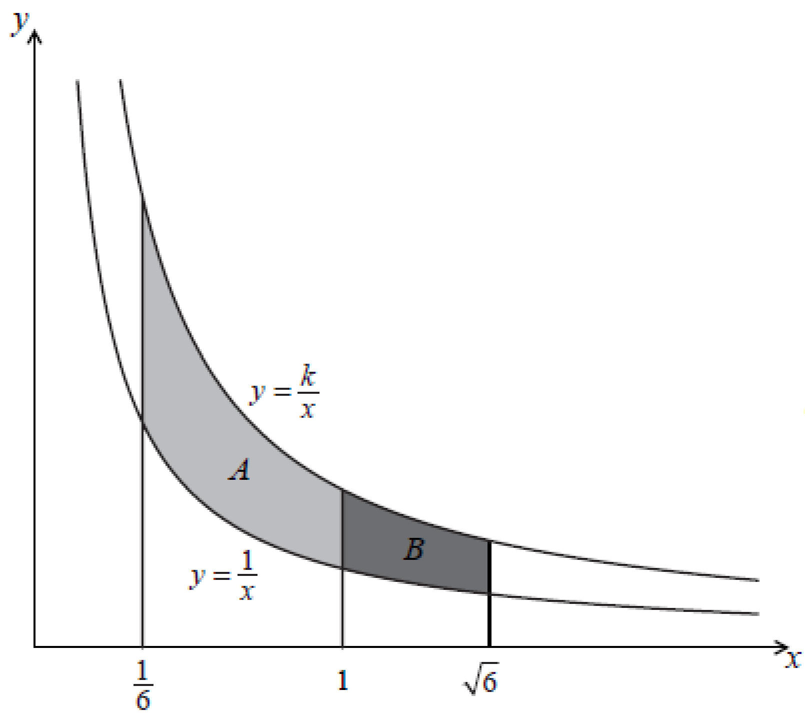
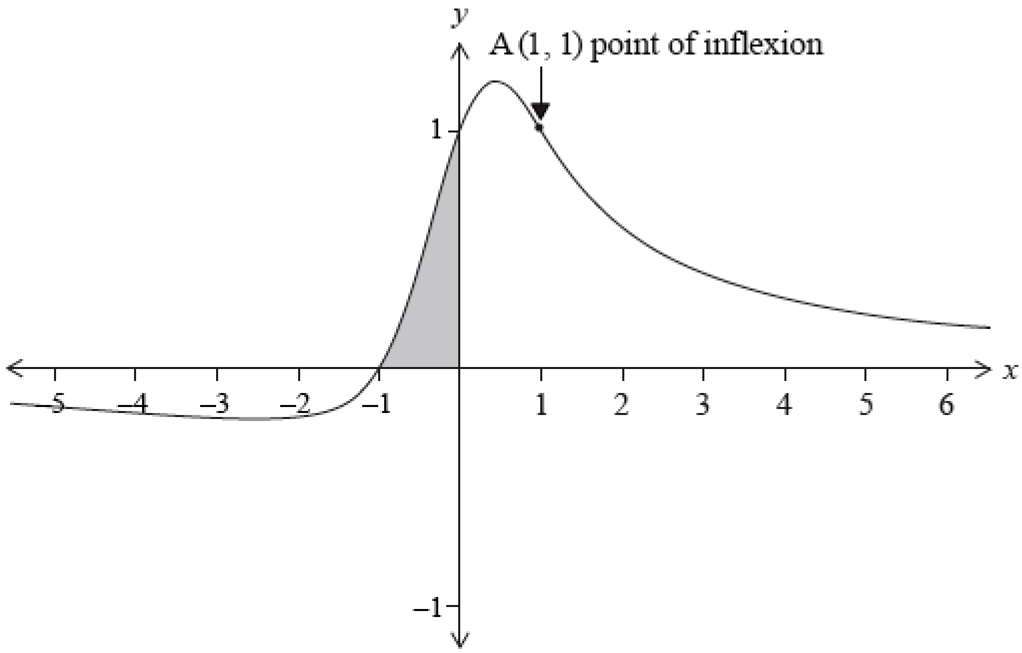


diagram not to scale

    a. Find the area of region A in terms of k .
    b. Find the area of region B in terms of k .
    c. Find the ratio of the area of region A to the area of region B
3. Consider the graph of $y = x + sin(x - 3), -\pi \le x \le \pi$
    a. Sketch the graph, clearly labelling the x and y intercepts with their values.
    b. Find the area of the region bounded by the graph and the x and y axes.

4. The graph of the function $f(x) = \frac{x+1}{x^2+1}$ is shown below.



The point (1, 1) is a point of inflexion. There are two other points of inflexion.

a. Find f'(x)

b. Hence find the x-coordinates of the points where the gradient of the graph of f is zero.

c. Find f''(x) expressing your answer in the form $\frac{p(x)}{(x^2+1)^3}$ where p(x) is a polynomial of degree 3

d. Find the x-coordinates of the other two points of inflexion.

e. Find the area of the shaded region. Express your answer in the form $\frac{\pi}{a} - ln\sqrt{b}$, where a and b are integers.

5. Find the area enclosed by the curve $y = arctanx$, the x-axis and the line $x = \sqrt{3}$.

- Volume of revolution

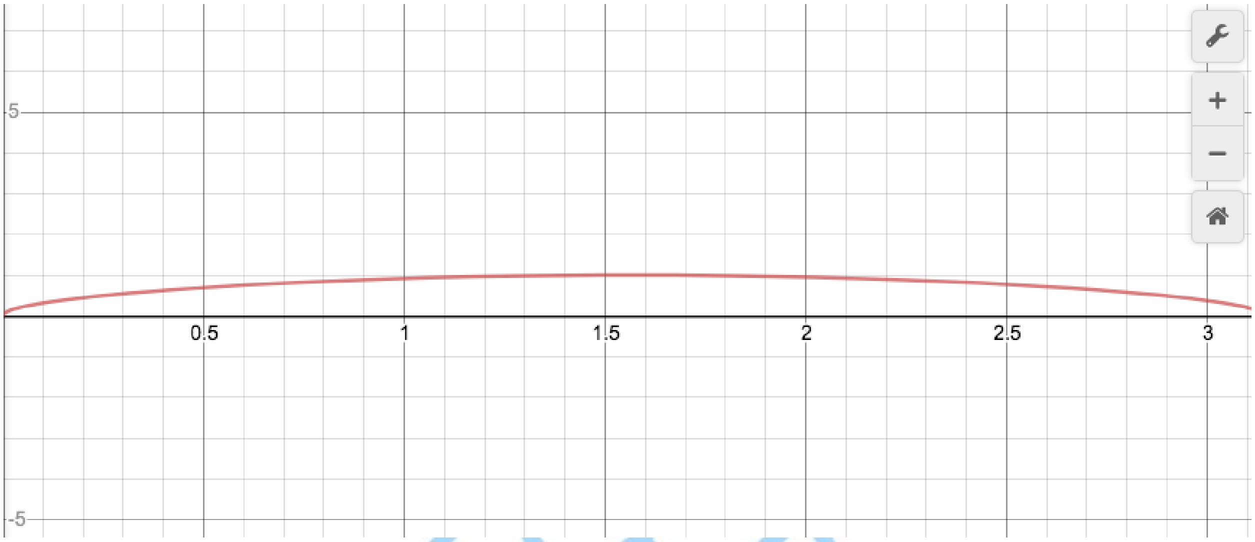
Rotating an equation around the y or x axis forms what is known as a solid of revolution. We can use integration to find the volume of this solid that forms between two intervals. The formula to find the volume of such a solid is given below.

Volume of revolution about the x or y-axes	$V = \int_a^b \pi y^2 \, dx$ or $V = \int_a^b \pi x^2 \, dy$

If the curve is rotated around the y-axis we use the first equation. Whereas if the curve is rotated around the x-axis we use the second formula provided.

Example:

Q. Find the volume of revolution when $y = \sqrt{\sin x}$ for $0 \leq x \leq \pi$is rotated 2πradians about the x-axis . The graph below is the graph of $y = \sqrt{\sin x}$ for $0 \leq x \leq \pi$.

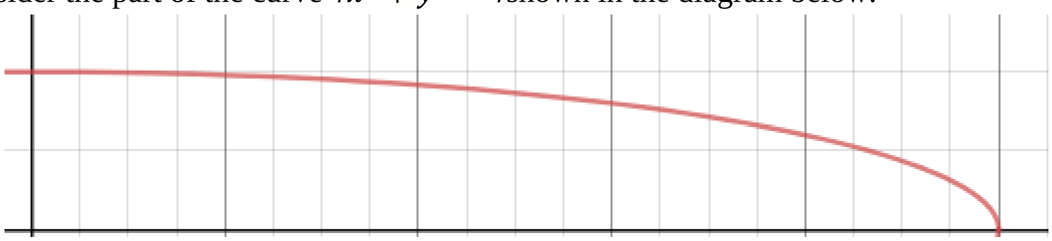


We need to use the formula to find the volume of the solid that forms when this curve rotates about the horizontal x-axis. Thus, $V = \pi \int_0^\pi (\sqrt{\sin x})^2 \, dx$.

$V = \pi(-\cos x)_0^\pi = \pi(-\cos \pi + \cos 0) = 2\pi.$

Example:

Q. Consider the part of the curve $4x^2 + y^2 = 4$shown in the diagram below.



The y-intercept is (0,2) and the x-intercept is (1,0).

a) Find an expression for $\frac{dy}{dx}$in terms of x and y.

Here we carry out implicit differentiation that we learnt earlier.

Therefore, $8x + 2y \cdot \left(\frac{dy}{dx}\right) = 0.$

$$\frac{dy}{dx} = -\frac{4x}{y}$$

b) A bowl is formed by rotating this curve through 2πradians about the x-axis. Calculate the volume of this bowl.

Since it rotated about the x-axis we use the formula $V = \pi \int_0^1 \quad y^2 dx = \pi \int_0^1 \quad (4 - 4x^2) \quad dx$

Therefore, $V = \frac{8\pi}{3}$.

Note:- If you are asked to find the volume between two curves the same formula applies except it can be modified to the following:-

$V = \pi \int_a^b \quad (y_U)^2 - (y_L)^2 \, dx$ (if the curves are rotated about the 'x' axis)

Exercise:

Q. The region enclosed between the curves $y = \sqrt{x}e^x$ and $y = e\sqrt{x}$ is rotated through 2π about the x-axis. Find the volume of the solid obtained.

Improper Integrals

The Math AA HL course only considers improper integrals that exist in the form $\int_a^\infty \quad f(x)\, dx$. We rewrite this as $\lim_{b\to\infty} \int_a^b \quad f(x)\, dx$. For this improper integral to exist, f(x) must approach 0 as x→ ∞.

Sample Question 1:

$\lim_{b\to\infty} \int_1^b \quad x^{-3}\, dx = \lim_{b\to\infty} [\frac{-1}{2x^2}]_1^b = \lim_{b\to\infty} (\frac{-1}{2b^2}) + \frac{1}{2}$. As b→ ∞, its reciprocal tends to zero. Thus the limit is 0.5.

Exercise:

1. Find the total area between $f(x) = e^{-x}(sinx)^2$ and the x-axis for $x \geq 0$.
2. Determine $\int_{ln\sqrt{3}}^{\infty} \quad \frac{1}{e^x + e^{-x}} \, dx$ using the substitution $u = e^x$
3. Find the total area between the function and the x-axis for $x \geq 0$
    a. $y = \frac{1}{1+x}$
    b. $y = 5^{-x}$
    c. $y = xe^{-x}$
    d. $y = xe^{-x^3}$

Kinematics

Kinematics is the branch of mechanics concerned with the motion of objects without reference to the forces which cause the motion. As mentioned earlier there are multiple applications of calculus and some of the most fundamental physics concepts have its roots in calculus. This application mainly deals with displacement, velocity and acceleration.

Displacement is a vector whose length is the shortest distance from the initial position to the final position. Velocity can be defined as the rate of change of displacement and therefore in terms of mathematics can be called the derivative of displacement. Acceleration is the rate of change of velocity which is therefore the derivative of the velocity function.

They are often denoted by the following:-

$$s = displacement$$
$$v = velocity$$
$$a = acceleration$$
$$t = time$$

Some important formulas include:-

1. $average\ velocity = \frac{s(t_2)-s(t_1)}{t_2-t_1}$
2. $instantaneous\ velocity = v(t) = s'(t)$
3. $change\ in\ displacement: \Delta s = s_2 - s_1$ or $\int_{t_1}^{t_2} v(t)\ dt$
4. $distance\ travelled = \int_{t_1}^{t_2} |v(t)|\ dt$
5. $average\ acceleration = \frac{v(t_2)-v(t_1)}{t_2-t_1}$
6. $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
7. $speed = |velocity|$

Note:- If $v(t)$ and $a(t)$ have the same sign then the speed is increasing at time 't'.
If $v(t)$ and $a(t)$ have opposite signs then the speed is decreasing at 't'.

$$s(t) \Rightarrow v(t) : \ you\ have\ to\ differentiate$$
$$v(t) \Rightarrow s(t) : you\ have\ to\ integrate$$
$$v(t) \Rightarrow a(t): you\ have\ to\ differentiate$$
$$a(t) \Rightarrow v(t) : \ you\ have\ to\ integrate$$

Example:

Q. A mass on a spring oscillates with displacement $s(t) = 8sin(8\pi t) + 6\ cm$ where $0 \leq t \leq 0.25s$.

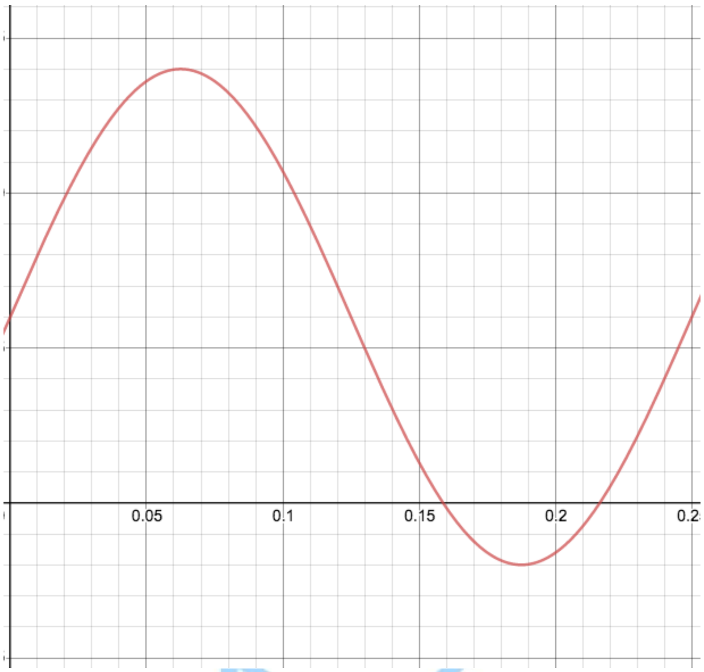
a) At what time is the displacement 6cm ?
b) Plot the graph of $s(t)$ for the given domain

c) Find where and when the mass changes direction

We begin by substituting 6 in place of s(t) to get the time at which displacement is 6 cm.

$$6 = 8sin(8\pi t) + 6$$
$$8sin(8\pi t) = 0$$
$$t = sin^{-1}(0)$$
$$8\pi t = 0, \pi, 2\pi\ldots$$
$$t = 0, \frac{1}{8}, \frac{1}{4}$$

Now we must plot the graph of the given function.



To find where the mass changes direction we need to observe the graph and mark its turning points. The graph has a maximum at t = $\frac{1}{16}$ $s$and a minimum at t = $\frac{3}{16}$ s, and therefore changes direction at those points.

IB Questions:

1. Particle A moves such that its velocity v ms^{-1}, at time t seconds, is given by $v(t) = \frac{t}{12+t^4}, t \geq 0$.Particle *B* moves such that its velocity v ms^{-1}, is related to its displacement s m by the equation $v(s) = arcsin(\sqrt{s})$
   a. Sketch the graph of v(t). Indicate clearly the local maximum and write down its coordinates.

   b. Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4} dt$.
   c. Find the exact distance travelled by particle A between t=0 and 6 seconds. Give your answer in the form karctan(b), k, b $\in R$
   d. Find the acceleration of particle B when s=0.1 m

2. A particle P moves in a straight line with displacement relative to origin given by $s = 2sin(\pi t) + sin(2\pi t), t \geq 0$, where t is the time in seconds and the displacement is measured in centimetres
   a. Write down the period of the function s
   b. Find expressions for the velocity, v and the acceleration a of P
   c. Determine all the solutions of the equation v=0, for $0 \leq t \leq 4$.
3. A particle moves in a straight line such that its velocity, v ms^{-1}, at time t seconds, is given by:

$$v(t) = \begin{cases} 5 - (t-2)^2, & 0 \leq t \leq 4 \\ 3 - \frac{t}{2}, & t > 4 \end{cases}.$$

   a. Find the value of t when the particle is instantaneously at rest.
   b. The particle returns to its initial position at t=T. Find the value of T
4. The particle P moves along the x-axis such that its velocity v ms^{-1} at time t seconds is given by $v = cos(t^2)$
   a. Given that P is at the origin O at time t=0, calculate
      i. the displacement of P from O after 3 seconds
      ii. the total distance travelled by P in the first 3 seconds
   b. Find the time at which the total distance travelled by P is 1 m.
5. A particle moves in a straight line, its velocity v ms^{-1} at time t seconds is given by $v = 9t - 3t^2, 0 \leq t \leq 5$. At t = 0, the displacement s of a particle from an origin O is 3m.
   a. Find the displacement of the particle when t=4.
   b. Sketch a displacement/time graph for the particle showing clearly where the curve meets the axes and the coordinates of the points where the displacement takes greatest and least values.
   c. For t>5, the displacement of the particle is given by $s = a \quad + bcos\frac{2\pi t}{5}$ such that s is continuous for all $0 \leq t$. Given further that s=16.5 when t=7.5, find a and b.
   d. For t>5, the displacement of the particle is given by $s = a \quad + bcos\frac{2\pi t}{5}$ such that s is continuous for all $0 \leq t$. Find the times t_1 and t_2 ($0<t_i<t_2<8$) when the particle returns to its starting point.

<u>Maclaurin Series</u>

A Maclaurin series is a power series that allows one to calculate an approximation of a function $f(x)$ for input values close to zero, given that one knows the values of the successive derivatives of the function at zero.

Maclaurin series	$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \ldots$
Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$
	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$
	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

Example:

If $f(x) = e^x \sin x$, find the first three terms using the Maclaurin Series formula.

$$f(x) = e^x \sin x$$
$$f'(x) = e^x(\sin x + \cos x)$$
$$f''(x) = e^x(-\sin x + \cos x) + e^x(\sin x + \cos x) = 2e^x \cos x$$
$$f'''(x) = 2e^x(-\sin x) + 2e^x \cos x \quad = 2e^x(\cos x - \sin x)$$

Now, we substitute zero in each of the equations.

$$f(0) = 0$$
$$f'(0) = 1$$
$$f''(0) = 2$$
$$f'''(0) = 2$$

Using the formula we can say that:

$$f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots..$$

Exercise:

Q. The function f is defined on the domain$]\frac{-\pi}{2}, \frac{\pi}{2}[$by f(x) = ln (1+sin x)

a) Show that $f''(x) = \frac{-1}{(1+\sin x)}$
b) Find the Maclaurin series for $f(x)$up to and including the term in x^4.

Q. Consider the functions f and g given by $f(x) = \frac{e^x+e^{-x}}{2}$and $g(x) = \frac{e^x-e^{-x}}{2}$.

a) Show that $f'(x) = g(x)$and $g'(x) = f(x)$
b) Find the first three non-zero terms in the Maclauric expansion of $f(x)$.

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