IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper F

Difficulty Rating: 4.045

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

If $f(x) = x^2 - 7x + 6$, solve the equation $f(x) = f(x+2)$. **(5)**

Question 2

The table below shows experimental data connecting two variables t and H.

t	5	10	20	40	50
H	4.1	8.5	18.0	42.0	50.0

It is assumed that t and H are related by an equation of the form

$$H = kt^n,$$

where k and n are non zero constants.

a) Find an equation of a straight line, in terms of well defined constants, in order to investigate the validity of this assumption. **(4)**

b) Plot a suitable graph to show that the assumption of part **(a)** is valid. **(3)**

c) Use the graph to estimate, correct to 2 significant figures, the value of k and the value of n. **(4)**

Question 3

A curve C has equation

$$y = x^2 + 8x + 12, \ x \in \mathbb{R}.$$

Describe fully a sequence of two transformations which map the graph of $y = x^2$ onto the graph of C. **(4)**

Question 4

A sequence x_1, x_2, x_3, x_4, … is given by

$$x_{n+1} = \frac{k - 5x_n}{x_n}, \quad x_1 = 1, \quad k > 5,$$

where k is a non zero constant.

a) Determine the value of x_3 in terms of k, giving the final answer as a single simplified fraction. **(3)**

It is further given that $x_3 > 6$.

b) Find the range of the values of k. **(4)**

Question 5

A circle with centre at the point C has equation

$$x^2 + y^2 - 10x - 6y + 14 = 0.$$

The straight line with equation $y = k$, where k is a non zero constant, is a tangent to this circle

a) Find the possible values k, giving the answers as exact simplified surds. **(5)**

The points A and B lie on the circumference of the circle and the point M is the midpoint of the chord AB.

b) Given the length of MC is 2, find the length of the chord AB. **(3)**

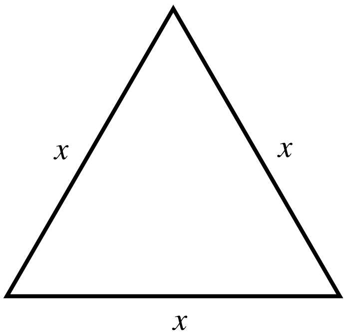
The straight line with equation

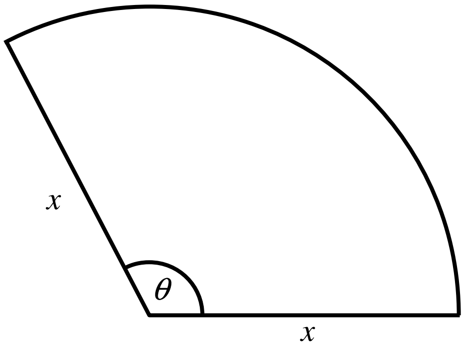
$$x - 2y - 9 = 0$$

is a tangent to the circle at the point D.

c) Determine the coordinates of D. **(5)**

Question 6





A wire of total length 60 cm is to be cut into two pieces. The first piece is bent to form an equilateral triangle of side length x cm and the second piece is bent to form a circular sector of radius x cm. The circular sector subtends an angle of θ radians at the centre.

a) Show that

$$x\theta = 60 - 5x \, . \qquad (2)$$

The **total** area of the **two** shapes is A cm^2.

b) Show clearly that

$$A = \frac{1}{4}\left(\sqrt{3} - 10\right)x^2 + 30x \, . \qquad (4)$$

c) Use differentiation to find the value of x for which A is stationary. **(5)**

d) Find, correct to three significant figures, the maximum value of A, justifying the fact that it is indeed the maximum value of A. **(3)**

Question 7

Find the possible solutions of the quadratic equation

$$x^2 + (k-1)x + k + 2 = 0 ,$$

where k is a constant, given that the equation has repeated roots. **(7)**

Question 8

The piecewise continuous function f is **even** with domain $x \in \mathbb{R}$.

It is defined by

$$f(x) \equiv \begin{cases} x^2 - 2x & 0 \le x \le 3 \\ 6 - x & x > 3 \end{cases}$$

a) Sketch the graph of f for all values of x. **(4)**

b) Solve the equation

$$f(x) = \frac{5}{4}$$ **(4)**

Question 9

Simplify the following logarithmic expression

$$\ln\left(2\sqrt{e}\right) - \frac{1}{3}\ln\left(\frac{8}{e^2}\right) - \ln\left(\frac{1}{3}e\right),$$

giving the answer in the form $\frac{1}{a} + \ln b$, where a and b are positive integers. **(6)**

Question 10

A function f is defined as

$$y = 3x^4 - 8x^3 - 6x^2 + 24x - 8, \; x \in \mathbb{R}, \; -2 \le x \le 3.$$

Sketch the graph of f, and hence state its range.

The sketch must include the coordinates of any stationary points and any intersections with the coordinate axes. **(14)**

Question 11

$$f(x) = \frac{3x+3}{x-2}, \ x \in \mathbb{R}, \ x \neq 2.$$

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of …

- … all the points where the curve meets the coordinate axes.
- … the equations of the two asymptotes of the curve. **(4)**

b) Solve the equation $f(x) = 2$. **(2)**

c) Hence solve the inequality $f(x) \geq 2$. **(2)**

Question 12

A curve C has equation

$$y = x - 2\ln\left(x^2+4\right), \ x \in \mathbb{R}.$$

a) Show clearly that

$$\frac{d^2y}{dx^2} = \frac{4\left(x^2-4\right)}{\left(x^2+4\right)^2}.$$ **(7)**

The curve has a single stationary point.

b) Find its exact coordinates and determine its nature. **(5)**

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Question 13

The sum of the first 25 terms of an arithmetic series is 1050 and its 25^{th} term is 72.

a) Find the first term and the common difference of the series. **(5)**

The n^{th} term of the series is denoted by u_n.

b) Given further that

$$117\left[\sum_{n=1}^{25} u_n - \sum_{n=1}^{k} u_n\right] = 233\sum_{n=1}^{k} u_n .$$

determine the value of k. **(7)**

Question 14

Determine, in exact simplified surd form, the solution pair (a,b) of the following simultaneous equations.

$$\sqrt{2}\ x+\sqrt{3}\ y=5 \qquad \text{and} \qquad \left(5\sqrt{3}-\sqrt{2}\right)x+\left(5\sqrt{2}-\sqrt{3}\right)y=10\sqrt{6} .$$

Detailed workings must be shown in this question. **(9)**

Question 15

The points A and B have coordinates $(8,2)$ and $(11,3)$, respectively.

The point C lies on the straight line with equation

$$y+x=14 .$$

Given further that the distance AC is twice as large as the distance AB, determine the two possible sets of coordinates of C. **(12)**

Question 16

$$\sin 2x \equiv \frac{2\tan x}{1+\tan^2 x}.$$

a) Prove the validity of the above trigonometric identity. **(3)**

b) Express $\frac{8}{(3t+1)(t+3)}$ into partial fractions. **(3)**

c) Hence use the substitution $t = \tan x$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x}\, dx = \ln 3.$$ **(8)**

Question 17

The curve C is given by the parametric equations

$$x = 3at,\ y = at^3,\ t \in \mathbb{R},$$

where a is a positive constant.

a) Show that an equation of the normal to C at the general point $(3at, at^3)$ is

$$yt^2 + x = 3at + at^5.$$ **(4)**

The normal to C at some point P, passes through the points with coordinates $(7,3)$ and $(-1,5)$.

b) Determine the coordinates of P. **(6)**

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Question 18

Solve the following trigonometric equation

$$\tan\left(\arctan 3x - \arctan 2\right) + \tan\left(\arctan 3 - \arctan 2x\right) = \frac{3}{8}. \qquad \textbf{(8)}$$

Question 19

A triangle has vertices at the points with coordinates $A(3,1)$, $B(7,4)$ and $C(10,-4)$.

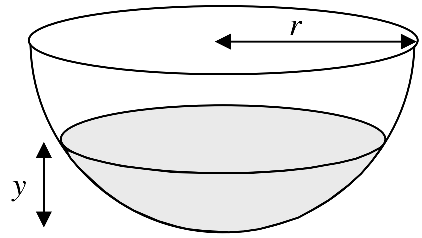
The acute angle θ is defined as the angle formed between AB and the straight line which is parallel to the y axis and passes through B.

Find the value of $\tan\theta$ and hence show that $\tan\left(\measuredangle ABC\right) = \frac{41}{12}$. **(8)**

Question 20

A water tank has the shape of a hollow inverted hemisphere of radius r cm.

The tank has a hole at the bottom which allows the water to drain out.



Let V, in cm^3, and y, in cm, be the volume and the height of the water in the tank, respectively, at time t seconds.

At time $t = 0$ the empty tank is placed under a running water tap. The rate at which the volume of the water in the tank is changing is proportional to the difference between the tank's constant diameter and the height of the water at that instant.

It can be shown by calculus that V and y are related by

$$V = \frac{1}{3}\pi\left(3ry^2 - y^3\right).$$

a) Show clearly that …

i. … $\dfrac{dy}{dt} = \dfrac{k}{\pi y}$,

where k is a positive constant. **(5)**

ii. … the time it takes to fill the tank is $\dfrac{\pi r^2}{2k}$ seconds. **(4)**

When the tank is full the running tap is instantly turned off but the water in the tank continues to leak out from the hole at the bottom.

b) Show it takes three times as long to empty the tank than it took to fill it up. **(9)**

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