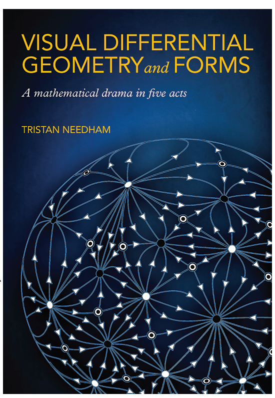
BOOK REVIEW

Visual Differential Geometry and Forms

A Mathematical Drama in Five Acts

Reviewed by Eric Poisson



Princeton University Press, Princeton, 2021.

Visual Differential Geometry and Forms: A Mathematical Drama in Five Acts
By Tristan Needham

Pick a squash from your fruit bowl, and stretch a string on its surface. With a pen, trace the curve under the string, and then remove the string. With a sharp knife, remove a narrow strip of peel around the marked path. Lay the strip flat on the kitchen counter. It will make a straight line. Magic? No! Geometry!

The field of differential geometry is a mature one, and as is typical in mathematics, with maturity comes a high degree of abstraction and formalization. Long forgotten are the meandering explorations of the pioneers, the intuitive reasoning, the contacts with the physical world. What emerges is a pristine work of platonic perfection, to be admired and passed on to posterity. For the poor unsophisticated physicist interested in learning this beautiful mathematics, it can be daunting to plough through 23 definitions, 52 propositions, 46 lemmas, and 17 theorems before coming to the first application of the formalism. The poor physicist (and, I suspect, a few secretive mathematicians) would welcome some motivation for the definitions and some plain-language explanation of the theorems. It is even possible, may I be forgiven for uttering such a blasphemous thought, that these individuals might be willing to trade rigour for a more intuitive understanding of the mathematics.

Tristan Needham, a professor of mathematics at the University of San Francisco, has worked very hard (incredibly hard!) to produce a presentation of differential geometry that will satisfy these individuals. The result is a truly unique textbook, and a remarkably successful one at that. There is really nothing like it out there. The author, it should be noted, previously applied his talents to a visual presentation of complex analysis [1].

The book presents the subject as an unfolding drama, in a succession of four acts (Act I: the nature of space, Act II: the metric, Act III: curvature, Act IV: parallel transport); the fifth act (Act V: forms) is a stand-alone introduction to differential forms. All this fills about 500 pages, complete with a multitude of beautifully crafted diagrams (about one per page), many practical exercises, and an annotated selection of additional reading materials.

There are three main factors that set this book apart from the rest of the textbook literature on differential geometry. The first is the liberal use of diagrams, which illustrate all aspects of a discussion or proof; this is where the book earns its "visual" attribute. The diagrams are sometimes pictures of fruit that were subjected to various experiments (toothpicks planted, strings strung, paths traced). Sometimes they are drawings that illustrate

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Communicated by Notices *Book Review Editor Katelynn Kochalski.*

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DOI: https://doi.org/10.1090/noti2574

mathematical constructions, reveal the meaning of symbols, and enhance intuition in a method of proof. Invariably the diagrams are beautiful to behold; they clearly result from a careful and deliberate design, and the author is an adept draughtsman.

The second factor is the Newtonian style of derivation—the geometrical style adopted by Newton in his *Principia*—adopted throughout the book. In the book's prologue, the author opens with a citation from geometer and Fields medalist Michael Atiyah, in which algebra is presented as the devil's tool, one which replaces geometrical thought by thoughtless calculation. This view thoroughly permeates the book: it is about beautiful visualization, not ugly computation. Everywhere the geometrical approach is favoured over the algebraic one. We still go through calculations—this is mathematics, after all—but each derivation emphasizes the geometry of the situation.

An example of the method, in the form of a geometrical proof of the identity $d\tan\theta/d\theta = 1 + \tan^2\theta$, is presented in the prologue. The proof introduces a small change of angle $\delta\theta$, shows the corresponding change in a right-angle triangle, constructs a similar triangle with a side length proportional to $\delta\theta$, and discovers what happens in the limit $\delta\theta \to 0$. To help with the proof, the author invents a new equality relation, $a \asymp b$, which means that if a and b depend on some ϵ, then $a/b \to 1$ when $\epsilon \to 0$. The notation is used throughout the book, and it allows the author to state cleanly what happens in the limit, without having to write $\lim_{\epsilon\to 0}$ every time. This proves to be very effective.

While I admire this style of presentation, and find it to be daring, unique, and original (in our times, if not in Newton's), I find that it is taken a little far at times. The book, for example, gives us ample opportunity to visualize and understand the meaning of the metric of a surface, but it does not give us a method to compute the metric for a given choice of embedded surface. This, of course, is consistent with the author's commitment to eschew the devil's tool, but it does create an imbalance. Perhaps a more balanced presentation of differential geometry would continue to emphasize the geometrical methods, but make some room for computations. (At the same time, given the near-universal emphasis on algebraic methods in the literature, the author should certainly be granted leave to go all in; balance will be achieved with a combination of texts.) Another indication of how far the author is willing to go is that Christoffel symbols are nowhere to be found in this book.

The third factor that makes this book unique is that it spares no effort to put the developments of the field in a historical context. We are introduced to the main players (Euler, Gauss, Jacobi, Riemann, Beltrami, Ricci-Curbastro, Poincaré, Levi-Civita, Hopf, Cartan) and their contributions to the field; sometimes details of their lifes are included. These historical tidbits add much richness to an already rich book, and make it a true delight to read.

The book's Act I (the nature of space) introduces the most fundamental idea of differential geometry, that contrary to Euclid's pronouncements, space does not have to be flat. Two alternative geometries are proposed, the spherical and the hyperbolic, and the Gaussian curvature is introduced in terms of the excess angle of a triangle.

These ideas are developed in Act II (the metric). The metric of a two-dimensional surface is introduced, and a formula to compute the Gaussian curvature is stated, anticipating a derivation to be presented in Act IV. (In the author's words, the formula appears like a phaser weapon from Star Trek, coming to us from the future.) The case of the pseudosphere is examined in detail, and described in various coordinates.

The book's climax (in the author's own estimation, and in mine as well) occurs in Act III (curvature). It begins with an exploration of the curvature of curves, and moves on to the (intrinsic and extrinsic) curvature of a surface. Gauss's famous *Theorema Egregium* follows, which relates the product of the radii of curvature to the Gaussian curvature. The second part of the act—a climax within the climax—is devoted to a thorough elucidation of the global Gauss-Bonnet theorem, a relation between the integrated curvature of a surface and its genus. The theorem is given three distinct proofs, based on different methods and ideas. The net outcome of all this is a firm and complete understanding of what the theorem means, with the different proofs providing complementary insights. This collection of chapters is a true masterpiece of clear and effective presentation.

Act IV (parallel transport) provides respite after the intensity of the previous act. A curious historical anomaly in the development of differential geometry is that the all-important notion of parallel transport was introduced (by Levi-Civita, in 1917) much after the defining contributions of Riemann and others. The author introduces this new tool, and wields it to derive the formula for the Gaussian curvature (the futuristic phaser weapon of Act II), as well as the Jabobi equation, which expresses the push of curvature on neighbouring geodesics. Parallel transport is also used to define the covariant derivative, and we finally arrive at Riemann's celebrated tensor. In the act's final chapter, the author could not resist (and who could blame him?) an excursion into Einstein's general theory of relativity. Here the apparatus of differential geometry finds a spectacular application in physics, and gravity is properly understood as a manifestation of the curved nature of our four-dimensional spacetime.

After a most satisfying evening at the theatre, the reader is invited to repair to the bar for a nightcap; and there, surprise!, Act 5 (forms) is about to unfold. The author's treatment of differential forms (completely antisymmetric tensors) is fairly standard: we start with 1-forms, move on to 2- and 3- and n-forms, are introduced to external differentiation, and then shown how all this relates to integration. As a bonus, we are presented with Cartan's powerful machinery for the effective computation of the Riemann tensor.

I have to confess that I was less convinced by this final portion of the book. A first reason has to do with a clash of objectives: while the rest of the book is all about visualization and geometrical insight, Act 5 is mostly about computation. The author has shockingly succumbed to the devil's tool! A second reason is that the presentation in Act 5 does not display the same flair and originality as in the rest of book; as I stated, it is fairly standard. As a third reason, I deplore the book's paucity of applications of this formalism, which is very powerful and applied to great benefit in many corners of theoretical physics. All in all, I feel that the reader might have been left free to go home after the play, to better digest the majestic four acts that preceded. A separate play, on a different night, might have served the topic better.

This quibble aside, the book offers a truly unique and original take on differential geometry, and it amply deserves inclusion within the pantheon of textbook deities. I very much admire the book, and I very much admire the author for having developed the subject in a thoroughly geometrical way. As stated, I do feel that the approach is sometimes taken a bit far; but I nevertheless appreciate the author's dogged determination.

Who will this book serve? In the prologue the author admits that he made no attempt to write the book as a classroom textbook, and indeed, this is not a typical text. I cannot imagine a novice student picking up this book and achieving mastery of differential geometry; to accomplish this one would require a better balance between visualization and computation. But a previously educated student, versed in the computational aspects of differential geometry, will be delighted by this book; the formulas will come to life, and a true and complete understanding will emerge. This student (who somehow seems a bit like me) will be well served indeed.

To conclude, let us return to the opening paragraph and the narrow strip of squash peel—the actual experiment is conducted in the book's Figure 1.11. The stretched string on the fruit's surface defines a curve of minimal length between two fixed points, a geodesic. The geodesic follows the surface's curvature, but geometrically, it is locally straight, in the sense that its tangent vector is parallel transported along the curve. When transplanted from the fruit to the kitchen counter, the locally straight curve is found to be globally straight. Geometry!

References

[1] Tristan Needham, *Visual complex analysis*, The Clarendon Press, Oxford University Press, New York, 1997. MR1446490

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Credits

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